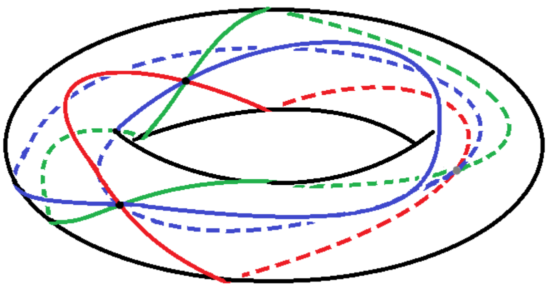
Universität Bremen

Diploma Thesis

Fundamental groups of toric Weyl arrangements

Sonja Riedel



Advisor: Dr. Emanuele Delucchi

July 2012

The figure on the title page shows the toric Weyl arrangement corresponding to the root system A_2.

Introduction

Toric arrangements are sets of hypersurfaces on the real or complex torus, that is to say on $(S^1)^n$ or $(\mathbb{C}^*)^n$. The hypersurfaces are defined by a character which is a map from the torus to the multiplicative groups S^1 respectively $\mathbb{C}^*$. They are the analogue to hyperplane arrangements, which are a set of hyperplanes in a vector space. The theory of toric arrangements is a relatively young and still developing field, the first attempts to this topic were made 1995 by Lehrer in [11]. It combines topology, combinatorics and algebra, as explained for example in the textbook [5] by DeConcini and Procesi.

Note that for every toric arrangement $\mathscr{A}$ there is a corresponding periodic hyperplane arrangement $\mathscr{A}^\uparrow$, which is a lifting of $\mathscr{A}$ as in [1]. The toric arrangement can be regarded as the orbit space of a suitable action (by translation via the characters) on the vector space containing $\mathscr{A}^\uparrow$.

On the other hand, a Weyl group W is a special kind of reflection groups (see [8]) and every reflection group yields an associated hyperplane arrangement, which is given by the mirrors of W. The *toric Weyl arrangement* are the ones corresponding to Weyl groups.

The aim of this thesis is the study of the fundamental group of the complement $\mathcal{M}(\mathscr{A})$ of toric Weyl arrangements. Our conjecture is that the set of generators can be reduced to one generator per hypersurface and the generators of the fundamental group of the torus.

For a hyperplane arrangement $\mathscr{A}^\uparrow$ we introduce its *Salvetti complex* $\mathcal{S}(\mathscr{A}^\uparrow)$, whose fundamental group is isomorphic to the fundamental group of the complement of $\mathscr{A}^\uparrow$ (as shown by Salvetti in [15]). By regarding paths on the 1-skeleton of this complex, we reduce the set of generators of $\pi_1(\mathcal{M}(\mathscr{A}^\uparrow)) = \pi_1(\mathcal{S}(\mathscr{A}^\uparrow))$ (as done by Salvetti in [15] and d'Antonio, Delucchi in [1]).

Then we turn to the fundamental group of the complement of the corresponding toric arrangement $\mathscr{A}$. In the process, we partially correct [1]. In order to do that we have to construct a fundamental domain of the action on the face poset of $\mathscr{A}^\uparrow$. Here too, more care than in [1] must be taken. Furthermore, we check our conjecture in the concrete cases with a toric Weyl arrangement corresponding to a Weyl group of rank 2.

In Chapter 1 delivers insight into the theory of reflections group and the main results about Weyl groups are recalled.

Chapter 2 contains the basics about hyperplane arrangements, including the construction of the Salvetti complex.

The construction of toric arrangements, their lifting to a hyperplane arrangement and the connection to Weyl groups is described in Chapter 3.

The consideration of the fundamental group of the complement of an arrangement starts in Chapter 4. We study the properties of the paths on the 1-skeleton of the Salvetti complex of a hyperplane arrangement $\mathscr{A}^{\uparrow}$ since they can be regarded as loops in $\mathcal{M}(\mathscr{A}^{\uparrow})$ around the hyperplanes. In the end, we translate this onto the torus and to the corresponding toric arrangement $\mathscr{A}$. We offer new or corrected proofs of some of the statements in [1].

In Chapter 5 we examine the construction of this fundamental domain suggested in [1], concluding that it is a fundamental domain depending on the choice of the base chamber C_0, the base point x_0, and the basis of the character lattice Λ.

We conclude with the examination of the fundamental groups of the complements of the toric Weyl arrangements corresponding to Weyl groups of rank 2.

Acknowledgements

I would like to thank all the people who helped with this thesis. In particular, I am very grateful for the support of my advisor Emanuele Delucchi, whose door was literally always open for me. Furthermore, I want to thank everyone who searched for mistakes in my thesis.

Finally, thanks to my family who supported me during my whole studies and taught me to believe in myself.

Contents

1 Reflection Groups

1.1 Reflections

In this chapter we recall the basics about finite reflection groups. For further knowledge see for example [8], for a better intuition of reflection groups consider [4].

Definition 1.1.1. *A* **reflection** *in an Euclidean space V endowed with a positive definite symmetric bilinear form (λ, μ) is an orthogonal transformation r that fixes a hyperplane H (subspace of codimension one) and sends any vector orthogonal to that hyperplane to its negative. The hyperplane H is called the* **mirror** *of r.*

The reflection is uniquely determined by the mirror, and vice versa. Thus any non-zero vector α determines a reflection, which we call r_α. The corresponding orthogonal hyperplane is denoted by H_α. Obviously, non-zero vectors proportional to α yield the same reflection. A formula for r_α:

$$r_\alpha(\lambda) = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha. \tag{1.1}$$

It is easy to see that all properties of a reflection are fulfilled:
The map r_α sends all λ in the span $\mathbb{R}\alpha$ of α to its negative and fixes H_α. Thereby (1.1) holds for all $\lambda \in V \cong \mathbb{R}\alpha \oplus H_\alpha$ since the right side of the formula is linear (with respect to λ). Obviously, the map $r_{c\alpha}$ equals r_α for any non-zero scalar c. A quick calculation shows that r_α is an orthogonal transformation, i.e., $(r_\alpha(\lambda), r_\alpha(\mu)) = (\lambda, \mu)$ for all $\lambda, \mu \in V$. Also, note that the map r_α is an involution.

A **finite reflection group** is a finite group generated by reflections.

1.2 Root systems

Since a reflection is uniquely determined by a vector, another approach to study reflection groups is to study corresponding sets of vectors.

Definition 1.2.1. *A finite set* Φ *of non-zero vectors in* V *is called a* **root system** *if it satisfies the following conditions:*

$$(R1) \qquad \Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\} \text{ for all } \alpha \in \Phi;$$
$$(R2) \qquad r_\alpha \Phi = \Phi \text{ for all } \alpha \in \Phi.$$

The elements of Φ *are called* **roots**.

Actually the groups generated by the reflections r_α for $\alpha \in \Phi$ are exactly the finite reflection groups. A proof of this can be found for example in [8]. In the following we will just discuss finite reflection groups. So keep in mind that all our reflection groups are finite, even though we omit mentioning "finite".

Now let us introduce the basic reflection groups A_n, B_n, C_n, D_n and the dihedral group $I_2(m)$. As in most literature we will use the same notation $A_n, B_n, C_n, D_n, I_2(m)$ for both the root system and the corresponding reflection group. Whether we are talking about the root system or the reflection group should be clear from the context.

Firstly, let us consider the reflection group $\mathbf{A_{n-1}}$ in $\mathbb{R}^n$ associated to the symmetric group S_n. A transposition (i, j) in S_n acts on $\mathbb{R}^n$ by permuting the i-th and j-th coordinate and thus defines a reflection in the hyperplane $\{x_i - x_j = 0\}$. Since S_n is generated by the transpositions, we obtain a reflection group isomorphic to the symmetric group, which has

$$A_n = \{e_i - e_j | i, j = 1, ..., n; i \neq j\}$$

as root system (where $\{e_1, ..., e_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$).

Secondly, there are the two distinct root systems $\mathbf{B_n}$ and $\mathbf{C_n}$, which generate the same reflection group (denoted by BC_n) in $\mathbb{R}^n$. These root systems are

$$B_n = \{\pm e_i \pm e_j, \pm e_i | i, j = 1, ..., n; i \neq j\} \quad \text{and}$$
$$C_n = \{\pm e_i \pm e_j, \pm 2e_i | i, j = 1, ..., n; i \neq j\}.$$

More concretely, we start with the same reflections as above, i.e., the ones associated to transpositions in S_n. Now we add the reflection obtained by sending e_i to its negative for all i. These sign changes generate a subgroup isomorphic to $\mathbb{Z}_2^n$, which intersects S_n trivially and is normalized by S_n, since conjugation just yields another sign change. Therefore, the reflection group BC_n is the semidirect product of S_n and the subgroup of sign changes.

The reflection group $\mathbf{D_n}$, $n \geq 4$, is a subgroup of index 2 of BC_n and consists of the sign changes which involve an even number of signs. Its root system is

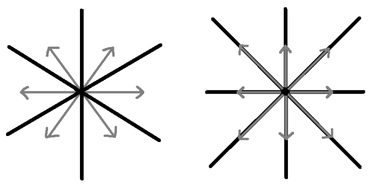
$$D_n = \{\pm e_i \pm e_j | i, j = 1, ..., n, i \neq j\},$$

which is of course a subsystem of B_n as well as C_n.

Furthermore, we have the dihedral group $\mathbf{I_2(m)}$ (in geometry often denoted as D_m), $m \geq 3$, which is the group of symmetries of a regular m-sided polygon in the Euclidean plane $\mathbb{R}^2$. It has order $2m$, and contains m rotations and m reflections. The subgroup of rotations is cyclic and generated by the rotation by $2\pi/m$. But this rotation by $2\pi/m$ can be obtained as a product of two reflections which meet at an angle of π/m. Hence the dihedral group is also a reflection group.

Figure 1.2.2.

Reflection groups



The reflection groups and root systems A_2 and B_2.

Definition 1.2.3. *If a reflection group W acts on V with no non-zero fixed points, then W is called* **essential** *on V.*

Proposition 1.2.4 (Compare [4], p.62)**.** *If Φ is a root system corresponding to the reflection group W, then the following are equivalent:*

(i) Φ spans V;

(ii) the intersection of all mirrors corresponding to Φ consists of one point;

(iii) W is essential on V.

Example 1.2.5. *The group A_{n-1} is not essential on $V = \mathbb{R}^n$, but A_{n-1} is essential on the hyperplane $\{x_1 + ... + x_n = 0\}$.*

From now on, we assume that our root system is essential. Otherwise, we will set $V = \text{span}(\Phi)$.

1.2.1 Positive and simple roots

Even though we can completely characterize reflection groups by root systems, there is still one disadvantage in using Φ for classification. The root system Φ may be extremely large compared to the dimension of V. Take for example the dihedral group in $\mathbb{R}^2$. Thus we look for a linear independent and generating subset of Φ.

Remember the definition of a **total order** on a vector space V. A transitive relation, denoted $<$, satisfying the following axioms:

1. For all $\lambda, \mu \in V$, exactly one of $\lambda < \mu$, $\lambda = \mu$, $\mu < \lambda$ holds.
2. For all $\lambda, \mu, \nu \in V$, if $\mu < \nu$, then $\lambda + \mu < \lambda + \nu$.
3. If $\mu < \nu$ and c is a non-zero scalar, then $c\mu < c\nu$ if $c >_{\mathbb{R}} 0$, while $c\nu < c\mu$ if $c <_{\mathbb{R}} 0$.

Thus, if there exists a linear function $f : V \to \mathbb{R}$ which does not vanish on Φ, i.e., $f(\alpha) \neq 0$ for all $\alpha \in \Phi$, then we get a total order on V by denoting $\lambda < \mu$ if $f(\lambda) <_{\mathbb{R}} f(\mu)$. Otherwise, we can construct a total order by taking an ordered basis $\lambda_1, ...\lambda_n$ of V and adopting the corresponding lexicographic order, i.e., $\sum a_i\lambda_i < \sum b_i\lambda_i$ means that $a_k < b_k$ where k is the least index i for which $a_i \neq b_i$.

Definition 1.2.6. *A root $\alpha \in \Phi$ is called* **positive** *relative to some total order on V if $0 < \alpha$. The subset Φ^+ of all positive roots is called* **positive system**. *Correspondingly, Φ^- is called the* **negative system**. *Due to $(R1)$ it is evident that $\Phi^- = -\Phi^+$ is exactly the subset of all negative roots (with respect to $<$).*

For example, the total order obtained by lexicographic order is constructed such that all elements λ_i of the basis are positive. Furthermore, it is easy to see that a root system Φ is the disjoint union of the positive and the negative system (rel. to some total order). Moreover, it is obvious that for a reflection group the positive system is not unique at all but is determined by the total order.

By now, we have decreased our amount of roots to the half, but the number of roots can still be extremely large compared to the dimension of the vector space. So we will use some linear algebra for reduction.

Definition 1.2.7. *A subset Π of Φ is called a* **simple system** *if Π forms a basis of the $\mathbb{R}$-span of Φ, and if all roots in Φ are a non-negative or a non-positive linear combination of the elements in Π.*

Clearly, the definition of simple roots highly depends on the current choice of the positive system Φ^+. For a proof of existence of simple systems see, e.g., [8, p.8]. Evidently, the simple system is depending on the choice of the positive system. There is actually a one-to-one correspondence between simple and positive systems.

Definition 1.2.8. *Let Φ be a root system, $W = \langle r_\alpha | \ \alpha \in \Phi \rangle$ the corresponding reflection group and $\Pi \subset \Phi$ any simple system. Then the* **rank of** W *is the cardinality of* Π*, which is the dimension of the span of Φ in V.*

Example 1.2.9. *A positive system of A_n is*

$$\Phi^+ = \{e_i - e_j | i, j = 1, ..., n+1, j < i\}$$

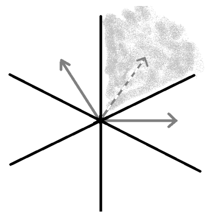
and the corresponding simple system is

$$\Pi = \{e_2 - e_1, ..., e_{n+1} - e_n\},$$

hence the rank of A_n is n.

Figure 1.2.10.

Positive and simple system.



The figure shows a positive system of the reflection group A_2. The dotted root is positive, but not simple. Moreover, the marked chamber is the corresponding Weyl chamber relative to this positive system (see Definition 1.3.9).

Example 1.2.11. *For B_n (respectively C_n) the set*

$$\Phi^+ = \{e_i, e_i \pm e_j | i, j = 1, ..., n; j < i\}$$

(respectively for C_n with $2e_i$ instead of e_i) is a positive system with the corresponding simple system

$$\Pi = \{e_1, e_2 - e_1, ..., e_n - e_{n-1}\}$$

(again $2e_1$ instead of e_1 for C_n), hence the rank of B_n (respectively C_n) is n, as well.

Proposition 1.2.12 (See Section 1.4 in [8]). *Let Π be a simple system, contained in the positive system Φ^+. If α is a simple root, then the reflection r_α will fix the positive system without α, i.e., the image $r_\alpha(\Phi^+\backslash\{\alpha\})$ equals $\Phi^+\backslash\{\alpha\}$.*

Proof. Let $\beta \in \Phi^+\backslash\{\alpha\}$, then β is a non-zero positive linear combination of the simple roots in the following way $\beta = \sum_{\gamma\in\Pi} c_\gamma\gamma$ and at least one $c_\gamma \neq 0$ with $\gamma \neq \alpha$ since α is the only multiple of α in Φ^+. Applying r_α, we get $r_\alpha\beta = \beta - c\alpha$ where the coefficients of $\gamma \in \Pi\backslash\{\alpha\}$ are still the same and positive. By the definition of simple roots, all coefficients are positive and thus $r_\alpha\beta \in \Phi^+$. We see at once that $r_\alpha\beta$ cannot be equal to α, because r_α maps $r_\alpha\beta$ to β and α to $-\alpha \in \Phi^-$. Therefore r_α maps $\gamma \in \Pi\backslash\{\alpha\}$ injectively into itself, and thus onto itself since we are talking about finite sets. □

Proposition 1.2.13 (See Section 1.2 in [8]). *Let t be an orthogonal transformation and α any non-zero vector in V, then $tr_\alpha t^{-1} = r_{t\alpha}$.*

Proof. Remember Definition 1.1.1 for $r_{t\alpha}$. We have to check that $tr_\alpha t^{-1}$ maps $t\alpha$ to its negative and that it fixes the hyperplane $H_{t\alpha}$. The first condition is evident. For the second we use that γ lies in H_α if and only if $t\gamma$ lies in $H_{t\alpha}$, because $(\gamma, \alpha) = (t\gamma, t\alpha)$ since t is orthogonal. Thus $tr_\alpha t^{-1}(t\gamma) = tr_\alpha(\gamma) = t\gamma$, whenever $\gamma \in H_\alpha$, and thereby $t\gamma \in H_{t\alpha}$. □

Especially this proposition says that $r_{w\alpha} = wr_\alpha w^{-1}$ lies also in W for $w \in W$. It also allows us to speak of conjugacy under $W = \langle r_\alpha |\ \alpha \in \Phi\rangle$ for roots in Φ, namely α, β are conjugate under W, if $\beta = w\alpha$, because that means $r_\beta = r_{w\alpha} = wr_\alpha w^{-1}$.

Theorem 1.2.14 (See Section 1.4 in [8]). *Any two positive (respectively simple) systems in Φ are conjugate under $W = \langle r_\alpha |\ \alpha \in \Phi\rangle$. That is to say, for two positive systems Φ^+ and Δ^+, there exists some $w \in W$ with $w\Phi^+ = \Delta^+$.*

Proof. Let Φ^+ and Δ^+ be two positive systems of the same root system Φ. Since a positive system contains exactly one root for each mirror in the reflection group, Φ^+ and Δ^+ have the same cardinality. We argue by induction on m, the cardinality of $\Phi^+\cap-\Delta^+$. If m is zero, the positive systems are obviously equal.

Now let m be greater than zero, then the simple system Π of Φ^+ cannot be fully contained in $-\Delta^+$, otherwise Φ^+ and Δ^+ would be equal. Hence there exists an $\alpha \in \Pi \cap -\Delta^+$. By Proposition 1.2.12 we know that the cardinality of $r_\alpha\Phi^+ \cap -\Delta^+$ equals $r - 1$. Since $r_\alpha\Phi^+$ is again a positive system, we can use our induction hypothesis. Consequently, there exists some $w \in W$ with $w(r_\alpha\Phi^+) = \Delta^+$. □

1.3 Coxeter and Weyl groups

1.3.1 Generators and relations

Now we will abstract the theory, so we want to move away from the concrete description of our group via reflections towards an algebraic description via generators and relations.

Definition 1.3.1. *If $\beta \in \Phi$ and $\beta = \sum_{\alpha\in\Pi} c_\alpha \alpha$ be its unique expression with either only non-negative or non-positive coefficients, then* $\mathrm{ht}(\beta) = \sum_{\alpha\in\Pi} c_\alpha$ *denotes the* **height** *of β.*

Theorem 1.3.2 (See Section 1.5 in [8])**.** *The group $W = \langle r_\alpha | \ \alpha \in \Phi\rangle$ is generated by the reflections r_α with $\alpha \in \Pi$. These reflections are called* **simple reflections***.*

Proof. Let W' denote the group generated by the simple reflections. We need to show that $W' \subset W$. Firstly, we will show that for an arbitrary positive root β the intersection of $W'\beta$ and Π is not empty. So let γ be the element with the smallest height in $W'\beta \cap \Phi^+$, which is non-empty since it contains at least β. Now we will show that γ is simple. So let $\gamma = \sum_{\alpha\in\Pi} c_\alpha \alpha$ be the expression in simple roots with non-negative coefficients, which exists since γ is a positive root. Thus $(\gamma, \gamma) = \sum_{\alpha\in\Pi} c_\alpha(\gamma, \alpha)$ being positive implies that $(\gamma, \alpha) > 0$ for some $\alpha \in \Pi$. If $\alpha = \gamma$ we are done.

Otherwise, we know that $r_\alpha\gamma$ is a positive root according to Proposition 1.2.12, hence $r_\alpha\gamma$ is in $W'\beta \cap \Phi^+$, too. On the other hand we deduce from Equation (1.1) that $r_\alpha\gamma = \gamma - \frac{2(\gamma,\alpha)}{(\alpha,\alpha)}\alpha$, in which the coeffient in front of α is positive, since $(\gamma, \alpha) > 0$ by construction. So we obtain $r_\alpha\gamma$ by subtracting a positive multiple of α from γ, though that would mean $\mathrm{ht}(r_\alpha\gamma) < \mathrm{ht}(\gamma)$, which contradicts the choice of γ. Thus $\gamma \in W'\beta \cap \Phi^+$ is simple and there exists $w \in W'$ with $w\beta = \gamma$, i.e., $W' \cap \Pi \neq \emptyset$.

Secondly, we show that $W'\Pi = \Phi$. Since β is an arbitrary positive root in the first part of this proof, we already know $\Phi^+ \subset W'\Pi$. Then $w\beta = \alpha$ is the same as $\beta = w^{-1}\alpha$ for appropriate $\alpha \in \Pi$ and $w \in W'$. If β is negative, then $-\beta \in \Phi^+$ implies that $-\beta = w\alpha$. Hence $\beta = wr_\alpha\alpha$, and again we get that β lies in $W'\Pi$. Therefore, $W'\Pi = \Phi$.

Finally, we are finally able to show $W' \subset W$. Let r_β be an arbitrary generator of W. Then there exists some $\alpha \in \Pi$ and some $w \in W'$ with $\beta = w\alpha$. Due to Proposition 1.2.13 it follows $r_\beta = wr_\alpha w^{-1} \in W'$. □

Definition 1.3.3. *We conclude from the above, that every $w \in W$ can be expressed as a combination of simple reflections, say $w = r_1...r_m$ with $r_i = r_{\alpha_i}$ for all $\alpha_i \in \Pi$. If m is minimal, then we call this expression* **reduced** *and $l(w) = m$ the* **length** *of w.*

The length of any w corresponds to the number of positive roots mapped to a negative root by w (see Sections 1.6-1.7 in [8]).

Definition 1.3.4. *A group F is called a* **free group** *generated by the set $S \subset F$ if every element can be written in a unique way as a reduced word in powers of generators $s \in S$, i.e., for all $g \in F$ we have $g = s_1^{a_1} ... s_m^{a_m}$ for some $s_1, ..., s_m \in S$, $a_1, ..., a_m \in \mathbb{Z}$. The empty word is the identity element* 1.

Furthermore, we know that we can express every reflection group as follows (compare with the Sections 1.6-1.9 in [8]):

Theorem 1.3.5. *The reflection group W with simple system $\Pi = \{\alpha_1, ..., \alpha_n\}$ is given by generators and relations*

$$W = \langle r_1, ..., r_n | (r_i r_j)^{m_{ij}} = 1 \rangle,$$

where $r_i = r_{\alpha_i}$ and m_{ij} is the order of $r_i r_j$.

We skip the proof, because it is not so important in order to understand the theory and uses a lot of combinatorics. The interested reader can find the proof in [8].

Definition 1.3.6. *Let $R = \{r_1, ..., r_n\}$ be a set of generators, thus in particular linearly independent. Then a* **Coxeter group** *is a group given by the generators and relations*

$$W = \langle r_1, ..., r_n | (r_i r_j)^{m_{ij}} = 1 \rangle,$$

where $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$. The value $m_{ij} = \infty$ is allowed and means no relation between r_i and r_j. The pair (W, R) is called a **Coxeter system**.

So a Coxeter group is an abstact group, which is completely determined by its generators and reflections. According to Theorem 1.3.5 every finite reflection group is a finite Coxeter group. Actually, the converse is also true, you can find a proof in Chapter 6 of [8].

1.3.2 Weyl groups

Definition 1.3.7. *A root system Φ is called* **crystallographic** *if it satisfies in addition to $(R1)$ and $(R2)$ the condition*

$$(R3) \qquad \frac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z} \text{ for all } \alpha, \beta \in \Phi.$$

These integers are called **Cartan integers**.

Actually, it is enough to require that the ratios have to be integers for all $\alpha, \beta \in \Pi$. In general, a subgroup G of $GL(V)$ is said to be crystallographic if it stabilizes a lattice L in V, thus a $\mathbb{Z}$-span of a basis of V.

Definition 1.3.8. *A reflection group W generated by a crystallographic root system is called* **Weyl group***.*

The reflection groups A_n, B_n, C_n and D_n are crystallographic, i.e., the roots systems are crystallographic and satisfy $(R3)$. So all of our examples for reflection groups are also Weyl groups except for the dihedral group, which in general is not crystallographic. Nonetheless, it is interesting to know that B_n and C_n are distinct crystallographic root systems with the same Weyl group BC_n (sometimes also B_n or C_n).

In fact, Weyl groups are exactly the reflection groups with m_{ij} equals 2,3,4 or 6 for all $i \neq j$ in Π (see [8, p.38]). This leads us to the exceptions, when $I_2(m)$ is a Weyl group, namely if m is either $3, 4$ or 6, since then the dihedral group coincides with Weyl groups. In the case m equals 3, this is A_2, in the case m equals 4 it coincides with BC_2 and $I_2(6)$ forms a Weyl group denoted by $\mathbf{G_2}$.

Given a root system Φ on V, let the sets $H_\alpha^+ = \{\lambda \in V | (\lambda, \alpha) > 0\}$ and $H_\alpha^- = \{\lambda \in V | (\lambda, \alpha) < 0\}$ denote the open half-spaces in which H_α divides V for every root α in Φ.

Definition 1.3.9. *Let Π be a positive system, then the hyperplanes H_α, for $\alpha \in \Pi$, cut the Euclidean space V into several connected components, which are called* **(Weyl) chambers***. The chamber determined by the intersection of the open half-spaces H_α^+ of the simple roots $\alpha \in \Pi$ is called the* **fundamental Weyl chamber** *(relative to Π).*

The Figure 1.2.10 shows the Weyl chamber (relative to the given positive system) of the reflection group A_2.

1.4 Coroots

Definition 1.4.1. *Let Φ denote a root system as above, then call*

$$\alpha^\vee := \frac{2\alpha}{(\alpha, \alpha)} \tag{1.2}$$

the **coroot** *to $\alpha \in \Phi$. The set of all coroots $\Phi^\vee$ is called the* **dual** *(or* **inverse***) of Φ.*

The set $\Phi^\vee$ is in fact also a root system and evidently produces the same corresponding Weyl group W as Φ. By definition, $(\alpha^\vee)^\vee = \alpha$ and Φ is the dual root system of $\Phi^\vee$.

Furthermore, in the majority of cases Φ and $\Phi^\vee$ are isomorphic, i.e., there exists a vector space isomorphism $\phi : V \to V$ mapping Φ to $\Phi^\vee$ such that $(\phi(\alpha), \phi(\beta)) = (\alpha, \beta)$ for all $\alpha, \beta \in \Phi$. However, the root systems B_n and C_n are dual to each other and produce the same Weyl group, but they are not isomorphic.

The coroots can also be considered as elements of the dual space V^*. A coroot $\alpha^\vee$ is then identified with the map

$$\alpha^\vee : V \to \mathbb{R}, \lambda \mapsto \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}.$$

For the reflection r_α it holds that $r_\alpha \lambda = \lambda - \alpha^\vee(\lambda)\alpha$ where we regard $\alpha^\vee$ as a linear map.

Definition 1.4.2. *The $\mathbb{Z}$-span $L(\Phi)$ of Φ in V is called the* **root lattice***, similar the* **coroot lattice** *$L(\Phi^\vee)$ is the $\mathbb{Z}$-span of $\Phi^\vee$.*

Both these lattices are stable under the action of W.

1.5 Affine Weyl groups

Until now we have just considered linear reflections, which are leaving the origin in V fixed. However, you can also examine **affine reflections**, whose reflecting hyperplane is not going through the origin. So we could just have required a reflection to be a non-identity isometry, that fixes an affine hyperplane and not necessarily the origin, instead of our Definition 1.1.1.

The theory with root systems corresponding to reflections makes only sense for orthogonal reflections in an Euclidean space. Nevertheless, we can work with affine reflections and root systems. We can construct affine reflections by using linear reflections and translations.

Definition 1.5.1. *For each root $\alpha \in \Phi$ and each $k \in \mathbb{Z}$, let*

$$H_{\alpha,k} = \{\lambda \in V | (\lambda, \alpha) = k\} \tag{1.3}$$

denote the **affine hyperplane** *which is defined by α and k. The corresponding affine reflection is defined by*

$$r_{\alpha,k}(\lambda) = \lambda - ((\lambda, \alpha) - k)\alpha^\vee. \tag{1.4}$$

Of course, we have $H_{\alpha,0} = H_\alpha$ as in Section 1.1 and $H_{-\alpha,-k} = H_{\alpha,k}$. Note that we obtain $H_{\alpha,k}$ by translating H_α by $\frac{k}{2}\alpha^\vee$, since for all λ in H_α the following holds

$$(\lambda + \tfrac{k}{2}\alpha^\vee, \alpha) = (\lambda, \alpha) + \tfrac{k}{2}(\alpha^\vee, \alpha) = \tfrac{k}{2}(\tfrac{2\alpha}{(\alpha,\alpha)}, \alpha) = k.$$

Let t_μ denote the translation by μ for an arbitrary $\mu \in V$, i.e., it sends λ to $\lambda + \mu$. Then

$$r_{\alpha,k}(\lambda) = \lambda - (\lambda, \alpha)\alpha^\vee + k\alpha^\vee = t_{k\alpha^\vee} \circ r_\alpha(\lambda),$$

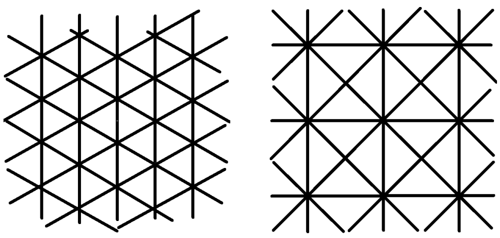
thus an affine reflection is the composition of a linear reflection and a translation.

Definition 1.5.2. *The group generated by all affine reflections $r_{\alpha,k}$ with $\alpha \in \Phi$ and $k \in \mathbb{Z}$ is called* **affine Weyl group** *and is denoted by $\widetilde{W}$.*

Proposition 1.5.3 (See Section 4.2 in [8])**.** *$\widetilde{W}$ is the semidirect product of W and the translation group corresponding to the coroot lattice $L(\Phi^\vee)$.*

This means especially that $\widetilde{W}/\Lambda$ is isomorphic to W, when $\Lambda = \langle\Phi^\vee\rangle_\mathbb{Z} = L(\Phi^\vee)$ acts on $\widetilde{W}$ by translation (compare [12, p.9]). Let us skip the proof and take a look at some figures to get a better intuition instead.

Figure 1.5.4. **Affine reflection groups.**



$\widetilde{A}_2$ *and* $\widetilde{B}_2$.

2 Hyperplane Arrangements

In this chapter we introduce some structure on hyperplane arrangements and we repeat some basics in algebraic topology. Then we will construct the Salvetti complex, which is homotopy equivalent to the complement of the corresponding hyperplane arrangement.

This chapter is orientated at the structure of the papers [1] and [2] by d'Antonio and Delucchi.

2.1 Basics

Definition 2.1.1. *Let V be a n-dimensional vector space over a field $\mathbb{K}$ (usually $\mathbb{R}$ or $\mathbb{C}$), and let $l_1, ..., l_m \in V^*$ be linear forms, and $b_1, ..., b_m \in \mathbb{K}$. Then we have m* **affine hyperplanes***, denoted by*

$$H_i = \{v \in V |\ l_i(v) = b_i\}.$$

The set containing all hyperplanes

$$\mathscr{A} = \{H_1, ..., H_m\}$$

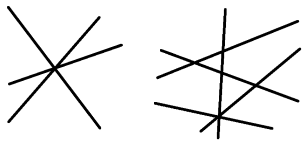
is called an **(affine) hyperplane arrangement***.*

Mostly we will just say "arrangement". An arrangement in the traditional definition as above is finite, but it is also possible to regard just **locally finite** arrangements, that is to say for all $v \in V$ only a finite number of hyperplanes in $\mathscr{A}$ contains v.

For example, the hyperplane arrangement corresponding to an affine Weyl group (see Example 1.5.4) is a locally finite arrangement, which is not finite. We will talk about finite arrangements, unless otherwise stated.

Definition 2.1.2. *If the intersection of all hyperplanes in $\mathscr{A}$ is empty, i.e., $\bigcap_{i=1}^m H_i = \emptyset$, the arrangement $\mathscr{A}$ is called* **centerless***. If the intersection is non-empty, i.e., $\bigcap_{i=1}^m H_i = M \neq \emptyset$, the arrangement is called* **centered** *with* **center** *M. In the case when all hyperplanes are linear, that is the same as to say M contains the origin, we will call the arrangement* **central***. Furthermore, it is called* **real** *or* **complex** *if V is a real or complex vector space.*

Figure 2.1.3. **Hyperplane arrangements.**



A centered and a centerless hyperplane arrangement.

An important part of the theory of hyperplane arrangements is the study of the complement of the arrangement, which we denote by

$$\mathcal{M}(\mathscr{A}) = V \backslash \Big(\bigcup_{i=1}^{m} H_i \Big). \tag{2.1}$$

Definition 2.1.4. *Let $\mathscr{A}$ be a real arrangement in the vector space V. The complement $\mathcal{M}(\mathscr{A})$ consists of several contractible connected components, which are called* **open chambers** *of $\mathscr{A}$. The set of open chambers will be denoted by $\mathcal{T}(\mathscr{A})$.*

Definition 2.1.5. *Let $\mathscr{A} = \{H_1, ..., H_m\}$ be an arrangement on V, and let $\mathscr{L}(\mathscr{A})$ be the set of non-empty intersections of hyperplanes in $\mathscr{A}$. Then we define a partial order on $\mathscr{L}(\mathscr{A})$ by reverse inclusion, i.e., $X \leq Y$ if and only if $Y \subseteq X$. Hence we get the* **intersection poset** *of $\mathscr{A}$ as*

$$\mathscr{L}(\mathscr{A}) = \big\{ \textstyle\bigcap_{i \in I} H_i |\ I \subset \{1, ..., m\} \big\} \backslash \{\emptyset\}$$

with minimal element V as the "trivial intersection" (thus $\bigcap_I H_i$ where $I = \emptyset$).

If the arrangement is centered, we get the center M as unique maximal element in $\mathscr{L}(\mathscr{A})$.

Definition 2.1.6. *Given a real arrangement $\mathscr{A}$, we can define the set of* **faces** *of $\mathscr{A}$ as*

$$\mathcal{F}(\mathscr{A}) := \{\overline{C} \cap X |\ C \in \mathcal{T}(\mathscr{A}), X \in \mathscr{L}(\mathscr{A})\}.$$

$\mathcal{F}(\mathscr{A}) = \mathcal{F}$ is called the **face poset** *of $\mathscr{A}$ partially ordered by inclusion.*

We will also refer to the maximal elements of $\mathcal{F}(\mathscr{A})$ as **(closed) chambers**.

A second way to construct the face poset is via the position of vectors with respect to the hyperplanes. Remember the construction of the open half spaces H_α^- and H_α^+ in Section 1.3.2. For a real arrangement of hyperplanes $\mathscr{A} = \{H_1, ..., H_m\}$ we have the **positive** and the **negative halfspaces**

$$H_i^+ = \{v \in V|\ l_i(v) > b_i\} \text{ and } H_i^- = \{v \in V|\ l_i(v) < b_i\}.$$

This leads us to the following definition:

Definition 2.1.7. *Two vectors $v, w \in V$ are* **similarly positioned** *with respect to $\mathscr{A}$ if they are on the same side of H_i for all i. On the same side of H_i, for $i = 1, ..., m$, means either both v and w lie in H_i, both lie in H_i^+ or both lie in H_i^-.*

Being similarly positioned with respect to $\mathscr{A}$ is obviously an equivalence relation and the equivalences classes are called faces. This construction is frequently used in literature to define a structure on V obtained by $\mathscr{A}$. Like in the first notation of chambers, the faces in this construction are open, but we want to work with the closed sets as faces.

As above in Definition 1.3.9 we distinguish one chamber:

Definition 2.1.8. *Let $B \in \mathcal{T}(\mathscr{A})$ be the chamber obtained as the intersection of all positive halfspaces, that is to say $B = \bigcap_{i=1}^m H_i^+$. Then the closure $\overline{B} \in \mathcal{F}(\mathscr{A})$ is called the* **base chamber**.

Definition 2.1.9. *If F is a face in $\mathcal{F}$, then every hyperplane $H \in \mathscr{A}$, which has a non-empty intersection with the interior F^o of F, contains F. Thus the intersection of all these hyperplanes also contains F and is an affine subspace, which we call the* **support** *of F and denote by* supp(F). *The* **dimension of** F *is the dimension of its support.*

Evidently, the dimension of chambers is equal to the dimension of V. Moreover, we denote the set of the faces of codimension i by $\mathcal{F}_i(\mathscr{A}) = \mathcal{F}_i$.

Definition 2.1.10. *A face of dimension $n - 1$ in the boundary of a chamber $C \in \mathcal{F}$ is called* **facet** *of C. A hyperplane containing a facet of C is a* **wall** *of the chamber C.*

2.2 Complexes

In order to understand the construction in the next section, we have to repeat some definitions of algebraic topology first. A complete introduction in the theory may be found in the book [7] by A. Hatcher.

Roughly speaking, a CW complex is a building obtained by glueing similar building blocks of increasing dimension, the k-cells, inductively together. Begin with the discrete set of points, the 0-cells, as the elements of smallest dimension. Then attach the k-cells of higher dimension along their boundaries inductively. The construction of a CW complex was introduced by J. H. C. Whitehead.

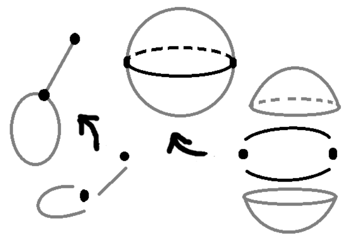
Definition 2.2.1. *For a positive integer k an* **(open)** k**-cell** *is a topological space, which is homeomorphic to the k-dimensional open ball, thus the interior of the k-disk D^k. Furthermore, the 0-cells are set to be points.*

Definition 2.2.2. *A* **cell complex** *or* **CW complex** *is a Hausdorff space X constructed in the following way:*

- *Start with the discrete set X^0 of 0-cells in X, the 0-skeleton.*
- *Then attach the cells of greater dimension inductively. The* k**-skeleton** *X^k is obtained from X^{k-1} by attaching k-cells e^k_α via maps $f_\alpha : S^{k-1} \to X^k$. This means that X^k is the quotient space of the disjoint union $X^{k-1} \amalg_\alpha D^k_\alpha$ with a collection of k-disks D^k_α under the identifications $x \sim f_\alpha(x)$ for all $x \in S^{k-1} = \partial D^k_\alpha$. Hence, the set $X^k = X^{k-1} \amalg_\alpha e^k_\alpha$ where each e^k_α is an open disk.*
- *Set $X = X^n$, if this process stops after some $n \in \mathbb{N}$. Then X is called n-dimensional, as the biggest cells in X. Otherwise, set $X = \bigcup_{k\in\mathbb{N}} X^k$ equipped with the weak topology, i.e., $A \subset X$ is open if and only if $A \cap X^k$ is open for all k, and call it infinite-dimensional.*

Figure 2.2.3.

CW complex.



The construction of a CW complex.

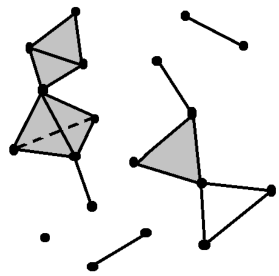
Definition 2.2.4. *A CW complex X is called* **regular** *if for each cell e^k_α the restriction of the attaching map $f_\alpha : \partial D^k \to f_\alpha(\partial D^k)$ is a homeomorphism.*

Definition 2.2.5. *Let $x_0, ..., x_k$ be affinely independent points in the Euclidean space $\mathbb{R}^n$, i.e., they do not lie in an affine subspace of dimension $k-1$. Then the convex hull of these $k+1$ points is a k-dimensional polytope, which is called a* k**-simplex** *and the points are the* **vertices** *of this simplex. Moreover, a non-empty subset of $\{x_0, ..., x_k\}$ spans a subsimplex, which is called* **face** *of the simplex.*

Definition 2.2.6. *A set of simplices* Δ *is a* **simplicial complex** *if every face of a simplex in* Δ *is a simplex in* Δ *as well and the intersection of any two simplices in* Δ *is a face of each of them.*

Figure 2.2.7.

Simplicial complex.

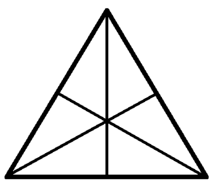


An 3-dimensional simplicial complex.

Barycentric subdivision is a standard way to divide an arbitrary convex polytope, or the cells of a cell complex, into simplices with same dimension. With this technique a cell complex can be regarded as a simplicial complex, since its cells are fragmented in simplices. In general, this works by fixing a "center of gravity", the **barycenter**, and then divide with the aid of the barycenter the polytope or cell into several simplices. For a k-simplex this are $(k+1)!$ smaller simplices. This descriptive definition should be enough for us, for a formal definition see [7]. So let us regard some pictures to get our intuition right.

Figure 2.2.8.

Barycentric subdivision.



The barycentric subdivision of a triangle.

Definition 2.2.9. *Let* X *be a topological space and* A *be a subspace of* X *with the inclusion map* $i : A \hookrightarrow X$. *Then* A *is called a* **deformation retract** *of* X *if there exists a continuous map* $r : X \to A$, *which is called* **retraction**, *such that* $r \circ i = id_A$ *and* $i \circ r \simeq_{id_A} id_X$.

Example 2.2.10. *The* 1*-sphere* S^1 *is a deformation retract of* $X = \mathbb{R}^2 \backslash \{0\} \simeq (\mathbb{C})^*$ *with the deformation retraction* $r : (\mathbb{C})^* \to S^1, z \mapsto \frac{z}{|z|}$.

The former definitions allow us to state the following proposition. The interested reader can find the proof in [7, p.36].

Proposition 2.2.11. *If* A *is a deformation retract of the topological space* X*, then the fundamental groups of* A *and* X *are isomorphic.*

2.3 Salvetti poset

Definition 2.3.1. *An arrangement* $\mathscr{A}$ *in* $V = \mathbb{C}^n$ *is called* **complexified** *if every hyperplane* H *in* $\mathscr{A}$ *is the complexification of a real hyperplane, i.e., the defining linear form* l *lies in* $\mathbb{R}^n \backslash \{0\}$ *and the defining scalar* b *is also real, such that*

$$H = \{z = x + iy \in \mathbb{C}^n |\ l(x) + il(y) = b\}.$$

Let the real part of a complexified arrangement be denoted by

$$\mathscr{A}_{\mathbb{R}} = \{H \cap \mathbb{R}^n |\ H \in \mathscr{A}\} = \{x \in \mathbb{R}^n |\ l_H(x) = b_H, H \in \mathscr{A}\}.$$

If $\mathscr{A}$ is a complexified arrangement, then $\mathscr{L}(\mathscr{A}) = \mathscr{L}(\mathscr{A}_{\mathbb{R}})$. Furthermore, we can use the combinatorical structure of $\mathscr{A}_{\mathbb{R}}$ to study the topology of $\mathscr{A}$. We will write $\mathcal{F}(\mathscr{A}) = \mathcal{F}(\mathscr{A}_{\mathbb{R}})$ and $\mathcal{T}(\mathscr{A}) = \mathcal{T}(\mathscr{A}_{\mathbb{R}})$.

Definition 2.3.2. *Let* $\mathscr{A}$ *be a complexified locally finite arrangement and* H^+ *and* H^- *open half spaces as above for all* H *in* $\mathscr{A}$*. Now we can define a* **sign vector** *for each face* $F \in \mathcal{F}_i$*,* $i \neq 0$*, as the function* $\eta_F : \mathscr{A} \to \{-, 0, +\}$*, such that*

$$\eta_F(H) := \begin{cases} +, & \text{if } F \subset H^+, \\ 0, & \text{if } F \subset H, \\ -, & \text{if } F \subset H^-. \end{cases} \tag{2.2}$$

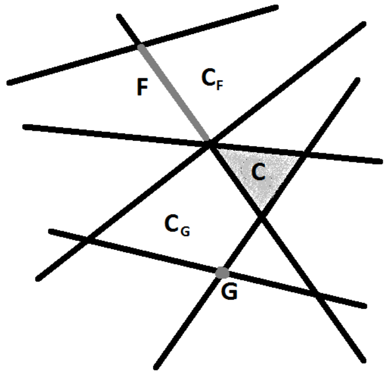
For $F \in \mathcal{F}_0$*, we just take the interior* $F^o \in \mathcal{T}(\mathscr{A})$ *instead of the closed chamber* F*. Then* $\eta_F(H) : \mathscr{A} \to \{+, -\}$ *can be defined as above with* F^o *in* H^+ *or* H^-*.*

Definition 2.3.3. *If $F \in \mathcal{F}(\mathscr{A})$ and C is a chamber in $\mathcal{F}(\mathscr{A})$ or $\mathcal{T}(\mathscr{A})$, let $C_F \in \mathcal{F}(\mathscr{A})$ denote the unique chamber such that*

$$\eta_{C_F}(H) = \begin{cases} \eta_F(H), & \text{if } \eta_F(H) \neq 0, \\ \eta_C(H), & \text{if } \eta_F(H) = 0. \end{cases}$$

Intuitively, C_F is the chamber touching F, which is closest to C.

Figure 2.3.4. **The chamber C_F.**



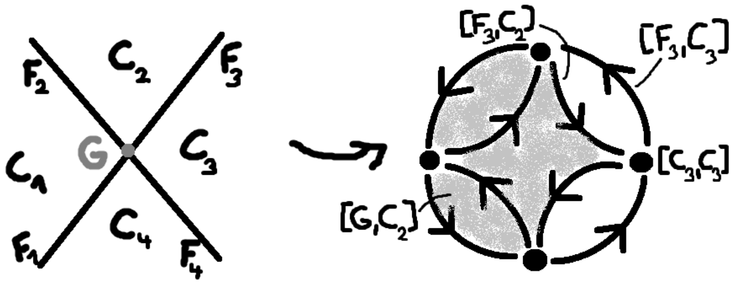
Definition 2.3.5. *Consider a complexified locally finite arrangement $\mathscr{A}$ and define the* **Salvetti poset** *as the set*

$$\mathrm{Sal}(\mathscr{A}) = \{[F, C] \mid F \in \mathcal{F}(\mathscr{A}), C \in \mathcal{F}_0, F \leq C\},$$

with the following order relation

$$[F_1, C_1] \leq [F_2, C_2] \Leftrightarrow F_2 \leq F_1 \text{ and } (C_2)_{F_1} = C_1.$$

Example 2.3.6. **Salvetti poset.**



The Salvetti poset of an hyperplane arrangement.

2.4 Salvetti complex

Definition 2.4.1. *Let $F \in \mathcal{F}_k$ and $H_{i_1}, ..., H_{i_l}$ be the hyperplanes containing it. Now let $C \in \mathcal{F}_0$ be a chamber which also contains the face F. Then* $\mathrm{op}(C, F)$ *denotes the* **opposite chamber** *of C with respect to F. Hence* $\mathrm{op}(C, F)$ *lies on the other side of H_{i_j} as C for all $j = 1, ..., l$.*

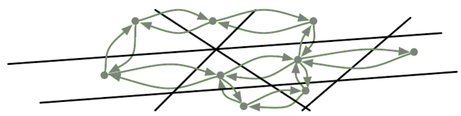
Now we are able to construct a graph $\mathcal{G}(\mathscr{A})$, which will be quite important for our considerations about the fundamental group of the complement of an affine arrangement in Chapter 4. In the Sections 2.4.7 and 2.4.8 we will see that the homotopy classes of loops in $\pi_1(\mathcal{M}(\mathscr{A}))$ can be regarded as paths in the graph $\mathcal{G}(\mathscr{A})$.

Definition 2.4.2. *Let $\mathscr{A}$ be a complexified locally finite arrangement, then set $\mathcal{G}(\mathscr{A})$ to be the graph which has $\mathcal{F}_0(\mathscr{A})$ as set of vertices and whose set of edges is given by*

$$E = \{e_{[F,C]} = (C, D) |\ F \in \mathcal{F}_1, C \in \mathcal{F}_0, F \leq C, \mathrm{op}(C, F) = D\}.$$

We say the edge $e_{[F,C]}$ *crosses* the hyperplane which supports F. Furthermore, we orient the edge $e_{[F,C]}$ from C to $\mathrm{op}(C, F)$, thus the index shows us the starting point and the faces and thereby the hyperplane which is crossed. Two chambers, C and C', are seperated by a hyperplane H, i.e., one lies in H^+ and the other one in H^-, if every path in $\mathcal{G}(\mathscr{A})$ connecting C and C' is crossing H.

Figure 2.4.3. **Example for the graph $\mathcal{G}(\mathscr{A})$.**



Reproduced by permission of E. Delucchi.

Definition 2.4.4. *A path in $\mathcal{G}(\mathscr{A})$ is called* **minimal** *if it crosses every hyperplane in $\mathscr{A}$ at most once. It is called* **positive** *if it follows the direction of all the edges. Otherwise it is called* **negative** *if it traverses all edges against their orientation.*

Definition 2.4.5 (Compare Definition 2.4 in [1])**.** *The* **unsubdivided Salvetti complex** *is the cell complex*

(i) whose 1-skeleton is the realisation of the graph $\mathcal{G}(\mathscr{A})$;

(ii) whose k-cells correspond to the pairs $[F, C]$ with $F \in \mathcal{F}_k$ and $C \in \mathcal{F}_0$ and

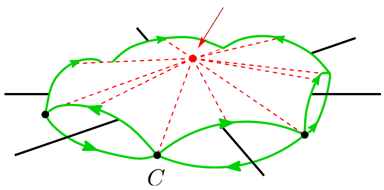
(iii) the 1*-skeleton of a k-cell $[F, C]$ is attached along the minimal positive directed paths in $\mathcal{G}(\mathscr{A})$ from C to* $\mathrm{op}(C, F)$*.*

A closer look at the condition (iii) shows that it states that a cell $[F_1, C_1]$ lies in the boundary of the cell $[F_2, C_2]$ in the unsubdivided Salvetti complex exactly when the relation $[F_1, C_1] \leq [F_2, C_2]$ holds. This means that the poset $\mathrm{Sal}(\mathscr{A})$ is the face poset of the unsubdivided Salvetti complex.

In addition the maximal elements of $\mathrm{Sal}(\mathscr{A})$ correspond to the pairs of a point and a chamber containing it, hence the set of maximal cells in the unsubdivided Salvetti complex is

$$\big\{[P, C] |\ P \in \mathcal{F}_n, C \in \mathcal{F}_0, P \leq C\big\}.$$

Figure 2.4.6. **Example of the Salvetti complex.**



Reproduced by permission of E. Delucchi.

The unsubdivided Salvetti complex is a regular cell complex and we obtain a simplicial complex $\mathcal{S} = \mathcal{S}(\mathscr{A})$ by taking its barycentric subdivision.

Definition 2.4.7. *Let $\mathscr{A}$ be a complexified locally finite arrangement in $\mathbb{C}^n$, then the simplicial complex $\mathcal{S} = \mathcal{S}(\mathscr{A})$ associated to $\mathrm{Sal}(\mathscr{A})$ is called the* **Salvetti complex** *of $\mathscr{A}$.*

$\mathcal{S}(\mathscr{A})$ is the same complex that Salvetti constructed in Part One of [15].

Theorem 2.4.8 (See Salvetti in Part One of [15])**.** *The Salvetti complex $\mathcal{S}(\mathscr{A})$ is a deformation retract of the arrangement's complement $\mathcal{M}(\mathscr{A}) = \mathbb{C}^n \setminus \bigcup_{i=1}^m H_i$. In particular, that means $\pi_1(\mathcal{M}(\mathscr{A})) = \pi_1(\mathcal{S}(\mathscr{A}))$ according to Proposition 2.2.11.*

The proof of this theorem is a little bit more complex, so the interested reader has to verify it by himself in Part One of [15] by M. Salvetti.

3 Toric Arrangements

Now we will transfer the theory of arrangements from vector spaces (or affine spaces) to tori. The structure of our spaces changes and it is more difficult to handle. In the first case we have simple linear algebra and in the second case we have to work with the algebraic geometry of tori. So let us recall some definitions, which for example can be found in [1] or [2].

Afterwards in Section 3.3 we will look at examples for toric Weyl arrangements, the construction of the toric arrangement corresponding to the affine Weyl group $\widetilde{A_2}$ and to the affine Weyl group $\widetilde{BC_2}$.

3.1 Definition

Definition 3.1.1. *The n-dimensional* **complex torus** *is the space $(\mathbb{C}^*)^n$ and the n-dimensional* **compact torus** *is $(S^1)^n$, with S^1 as the unit circle in $\mathbb{C}$.*

Definition 3.1.2. *Let $T = X^n$ be the n-dimensional compact or complex torus, thus X is either S^1 or $\mathbb{C}^*$. Then the maps $\chi : T \to X$ given by the Laurent monomials over X are the* **characters** *of T, thus we have*

$$\chi(x) = x_1^{a_1} \dots x_n^{a_n} \text{ with } a = (a_1, ..., a_n) \in \mathbb{Z}^n, \text{ for all } x \in T.$$

The set of all characters of T will be denoted by Λ. It is a lattice with pointwise multiplication as operation, which is isomorphic to $\mathbb{Z}^n$ via the mapping $a \mapsto x_1^{a_1} \dots x_n^{a_n}$.

Definition 3.1.3. *Given a compact or complex torus T and its set of characters Λ, then the set*

$$H_{\chi,a} = \{x \in T | \chi(x) = a\} \text{ with } \chi \in \Lambda, a \in S^1 \text{ or } a \in \mathbb{C}^*$$

is a **hypersurface** *of T.*

Definition 3.1.4. *Let A be a finite subset of $\Lambda \times \mathbb{C}^*$, a* **(complex) toric arrangement** *$\mathscr{A}$ is the collection of hypersurfaces generated by A, i.e.,*

$$\mathscr{A} = \{H_{\chi,a} | (\chi, a) \in A\}.$$

We will also write

$$\mathscr{A} = \{(\chi, a) | \chi \in \Lambda, a \in \mathbb{C}^*\}$$

and may think of $\mathscr{A}$ *as the finite collection of the hypersurfaces* $H_{\chi,a}$*. The complement of* $\mathscr{A}$ *is*

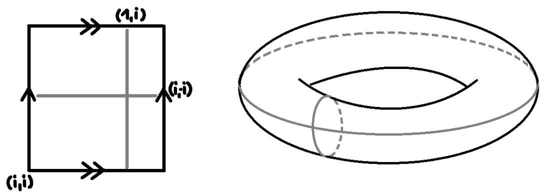
$$\mathcal{M}(\mathscr{A}) = (\mathbb{C}^*)^n \backslash \bigcup_{(\chi,a)\in A} H_{\chi,a}.$$

Definition 3.1.5. *If* A *is a finite subset of* $\Lambda \times S^1$ *and* Λ *a finitely generated lattice as above, then a* **real toric arrangement** *is given by the collection of hypersurfaces*

$$H^{\mathbb{R}}_{\chi,a} = \{x \in (S^1)^n | \chi(x) = a\} \textit{ with } (\chi, a) \in A.$$

As in Definition 2.3.1, when a complex toric arrangement restricts to a real toric arrangement on $(S^1)^n$*, we call it* **complexified**.

Example 3.1.6. A simple example of a toric arrangement.



The toric arrangement on the 2-dimensional compact torus which is given by the characters $t = 1$ *and* $s = -i$.

Instead of the former concrete definition of the torus and its lattice, we can also introduce a toric arrangement in a more abstract way, starting with a finitely generated lattice as basic object rather than the "concrete" torus.

Definition 3.1.7. *Let* $\Lambda \cong \mathbb{Z}^n$ *be a finitely generated lattice, then we define the corresponding* **complex torus** *to be*

$$T_\Lambda = \mathrm{Hom}_{\mathbb{Z}}(\Lambda; \mathbb{C}^*).$$

Similarly,

$$T^{\mathbb{R}}_\Lambda = \mathrm{Hom}_{\mathbb{Z}}(\Lambda; S^1)$$

is the corresponding **compact torus**. *For a choice of a basis* $\{\chi_1, ..., \chi_n\}$ *of* Λ *we get the isomorphisms*

$$\varphi : T_\Lambda \to (\mathbb{C}^*)^n \textit{ with } g \mapsto (g(\chi_1), ..., g(\chi_n)),$$

$$\varphi : T_\Lambda^{\mathbb{R}} \to S^1 \text{ with } g \mapsto (g(\chi_1), ..., g(\chi_n)).$$

Remark 3.1.8. *The character lattice of T_Λ is naturally isomorphic to Λ (see Remark 13 in [2]), therefore we can identify them in the following.*

Based on this definition, we can construct a toric arrangement as above.

In contrast to an affine arrangement, the hypersurfaces in a toric arrangement are not necessarily connected, such as in Example 3.1.6. Even more, the intersection of a finite collection of connected hypersurfaces does not have to be connected in general. Thus we need another combinatorial invariant to study the topology of the complement $\mathcal{M}(\mathscr{A})$ in the toric case corresponding to the intersection poset (see Definition 2.1.5) in the affine case.

Definition 3.1.9. *Let $\mathscr{A}$ be a toric arrangement on T_Λ. Then we consider the set $\mathcal{C}(\mathscr{A})$ of the connected components of non-empty intersections of hypersurfaces in $\mathscr{A}$. The elements in $\mathcal{C}(\mathscr{A})$ are* **layers** *of $\mathscr{A}$, and $\mathcal{C}(\mathscr{A})$, ordered by reverse inclusion, is the* **layer poset** *of $\mathscr{A}$.*

In the same way in comparison to the chambers (see Definition 2.1.4) and the faces (see Definition 2.1.6) in a affine hyperplane arrangement, we define the toric ones.

Definition 3.1.10. *Given a complexified toric arrangement $\mathscr{A}$, let $\mathscr{A}^{\mathbb{R}}$ denote the arrangement of hypersurfaces on the real torus $T_\Lambda^{\mathbb{R}}$. Then the* **chambers** *of $\mathscr{A}$ are the connected components of $\mathcal{M}(\mathscr{A}^{\mathbb{R}}) = T_\Lambda^{\mathbb{R}} \setminus \bigcup H_{\chi,\alpha}^{\mathbb{R}}$. $\mathcal{T}(\mathscr{A})$ denotes the set of all chambers of $\mathscr{A}$. The set of* **faces** *of $\mathscr{A}$ is defined as*

$$\mathcal{F}(\mathscr{A}) := \{\overline{C} \cap X \mid C \in \mathcal{T}(\mathscr{A}), X \in \mathcal{C}(\mathscr{A})\}.$$

As above, $\mathcal{F}_i$ is the subset of $\mathcal{F}(\mathscr{A})$ containing all faces of codimension i.

The faces of $\mathscr{A}$ are the cells of a cell complex as in the affine case.

Definition 3.1.11. *A toric arrangement is called* **essential** *if the layers of maximal codimension are points.*

Unless otherwise stated, our arrangement $\mathscr{A}$ will be essential and complexified from now on. Since there always exists an essentialisation for all toric arrangement (see Remark 3.6 in [1]), it is no restriction to consider only essential arrangements.

3.2 Covering space

In this section, we will see the connection between toric arrangements and hyperplane arrangements.

Given a lattice Λ of rank n, consider the covering map

$$p : \mathbb{C}^n \cong \mathrm{Hom}_{\mathbb{Z}}(\Lambda;\mathbb{C}) \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda;\mathbb{C}^*) = T_\Lambda, \qquad (3.1)$$
$$g \mapsto \exp \circ g,$$

where $\exp : \mathbb{C} \to \mathbb{C}^*$, $z \mapsto e^{2\pi i z}$, is the exponential map. Since we can identify $\mathrm{Hom}_{\mathbb{Z}}(\Lambda;\mathbb{C})$ with $\mathbb{C}^n$, p is the universal covering map

$$(x_1, ..., x_n) \mapsto (e^{2\pi i x_1}, ..., e^{2\pi i x_n})$$

of the torus T_Λ. Moreover, we get a restriction of p on the compact torus

$$\mathbb{R}^n \cong \mathrm{Hom}_{\mathbb{Z}}(\Lambda;\mathbb{R}) \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda;S^1) = T_\Lambda^{\mathbb{R}}. \qquad (3.2)$$

Thus we get an associated periodic affine hyperplane arrangement in $\mathbb{C}^n \cong \mathrm{Hom}_{\mathbb{Z}}(\Lambda;\mathbb{C})$ for the toric arrangement $\mathscr{A}$. This hyperplane arrangement is not finite, but locally finite. Besides it is the preimage of $\mathscr{A}$ under p and we will denote it by

$$\mathscr{A}^{\uparrow} = \{(\chi, z) \in \Lambda \times \mathbb{C} \,|\, (\chi, e^{2\pi i z}) \in \mathscr{A}\}. \qquad (3.3)$$

The upwards arrow should indicate that $\mathscr{A}^{\uparrow}$ is obtained by lifting our original space. An example of this connection between toric and affine arrangements can be found in the next section.

Remark 3.2.1 (See [2], Remark 18)**.** *If $\mathscr{A}$ is complexified, so is $\mathscr{A}^{\uparrow}$.*

Since $\mathscr{A}^{\uparrow}$ is a locally finite complexified hyperplane arrangement, there exists a corresponding Salvetti complex, which we denote by $\mathcal{S}^{\uparrow} = \mathcal{S}^{\uparrow}(\mathscr{A}) := \mathcal{S}(\mathscr{A}^{\uparrow})$.

The character lattice Λ acts by translation cellulary on $\mathcal{S}^{\uparrow}$ and continuously on the covering space $\mathcal{M}(\mathscr{A}^{\uparrow})$. Thus we can consider the orbit space $\mathcal{S}^{\uparrow}/\Lambda$.

Proposition 3.2.2 (See [12], Lemma 1.1)**.** *Let $\mathscr{A}$ be an toric arrangement and Λ its character lattice, then the embedding $\mathcal{S}^{\uparrow} \to \mathcal{M}(\mathscr{A}^{\uparrow})$ induces an embedding $\mathcal{S}^{\uparrow}/\Lambda \to \mathcal{M}(\mathscr{A})$, such that the quotient $\mathcal{S}^{\uparrow}/\Lambda$ is a deformation retract of $\mathcal{M}(\mathscr{A})$.*

3.3 And Weyl groups?

In this section we will study toric arrangements corresponding to Weyl groups, which we call **toric Weyl arrangements.**

Let V be an Euclidean space isomorphic to $\mathbb{R}^n$, Φ an essential crystallographic root system in V and $\widetilde{W}$ the corresponding affine Weyl group. Then let $\Lambda = \langle \Phi^\vee \rangle_\mathbb{Z}$ be the coroot lattice as in Section 1.5, which is isomorphic to $\mathbb{Z}^n$, since Φ is essential. Now we construct a torus as the quotient $T = V/\Lambda \cong \mathbb{R}^n/\mathbb{Z}^n = (S^1)^n$. Taking the quotient over Λ makes sense, because $\widetilde{W}/\Lambda$ is isomorphic to W (see Proposition 1.5.3).

Furthermore, let $\mathscr{A}^\uparrow = \{H_{\alpha,k} \,|\, \alpha \in \Phi, k \in \mathbb{Z}\}$ be the affine arrangement which corresponds to $\widetilde{W}$. The hypersurfaces in the obtained toric arrangement are exactly the orbits of the hyperplanes in $\mathscr{A}^\uparrow$. Additionally, $\overline{H_{\alpha,k}}$ and $\overline{H_{\alpha,m}}$ are equal if and only if $\overline{H_{\alpha,m}}$ is a translation of $\overline{H_{\alpha,k}}$ by a coroot. Particularly, this means that there are at most two orbits containing hyperplanes orthogonal to a fixed root $\alpha \in \Phi$, since according to Section 1.5 the following holds

$$H_{\alpha,k} = t_{\frac{k}{2}\alpha^\vee}(H_\alpha) = H_\alpha + \frac{k}{2}\alpha^\vee.$$

There exists exactly one hypersurface if

$$H_{\alpha,1} = t_{\beta^\vee}(H_\alpha) = H_\alpha + \beta^\vee$$

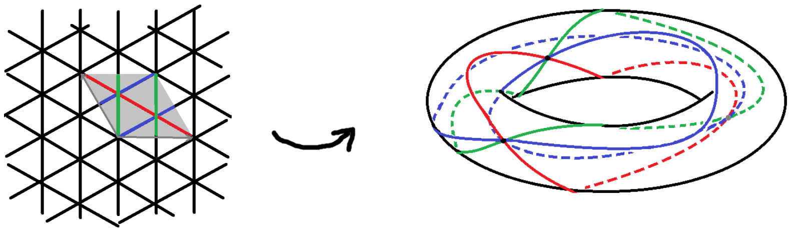
for some $\beta^\vee \in \Phi^\vee$ (obviously distinct to $\alpha^\vee$). Hence,

$$\mathscr{A} = \{\overline{H_{\alpha_i,k_i}} | \alpha_i \in \Phi, k_i \in \mathbb{Z}_2\}$$

is the toric arrangement associated to the affine Weyl group $\widetilde{W}$.

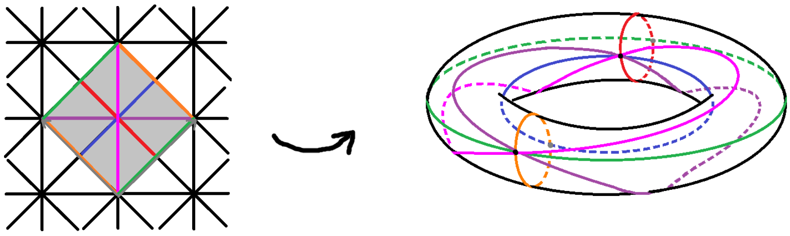
For a better understanding we will take a look at two examples. First, the construction of the toric arrangement associated to the affine Weyl groups $\widetilde{A_2}$. Afterwards we construct the toric arrangements obtained by starting with the root system B_2

Figure 3.3.1. **The toric Weyl arrangement $\mathscr{A}_2$.**



The figure shows the affine arrangement $\mathscr{A}_2^{\uparrow}$ with fundamental region of the action of Λ on $\mathbb{R}^2$ and the corresponding toric Weyl arrangement $\mathscr{A}_2$.

Figure 3.3.2. **The toric Weyl arrangement of BC_2.**



The figure shows the affine arrangement corresponding to $\widetilde{B_2}$ with fundamental region of the action of Λ on $\mathbb{R}^2$ and the corresponding toric Weyl arrangement.

4 Fundamental Group

In this chapter we want to study the fundamental group of the complement of a complexified toric arrangement. In order to do this, we have to take a look at the fundamental group of the complement of the associated affine arrangement first.

So we start by recalling the presentation of the fundamental group of an affine arrangement given by Salvetti in [15]. Then we transfer the gained achievements to the toric arrangement, which we have considered at the beginning, as Delucchi and d'Antonio did in [1].

4.1 The affine case

Let again $\mathscr{A}$ be a complexified toric arrangement, $\mathscr{A}^{\uparrow}$ be its lifting and $\mathcal{F}(\mathscr{A}) = \mathcal{F}$ respectively $\mathcal{F}(\mathscr{A}^{\uparrow}) = \mathcal{F}^{\uparrow}$ be their face posets. Furthermore let $\mathcal{F}_i$ respectively $\mathcal{F}_i^{\uparrow}$ denote the set of faces of codimension i. Now we can study the paths on the Salvetti complex $\mathcal{S}^{\uparrow}$, i.e., in $\mathcal{M}(\mathscr{A}^{\uparrow})$ according to Theorem 2.4.8, by regarding the paths along the edges of the graph $\mathcal{G}^{\uparrow} = \mathcal{G}(\mathscr{A}^{\uparrow})$ (introduced in the Definition 2.4.2).

4.1.1 Paths on $\mathcal{G}(\mathscr{A}^{\uparrow})$

We index the edges of $\mathcal{G}^{\uparrow}$ by the face of codimension one which they are crossing. Moreover, for $F \in \mathcal{F}_1$ the notation l_F means crossing F along the direction of the edge, l_F^{-1} against the direction of the edge. Fixing the start point of a path determines the edges which are used and in particular, in which direction. So there is no confusion, which way we are going.

Definition 4.1.1. *A path ν on $\mathcal{G}^{\uparrow}$ is* **positive** *if there exist $F_1, ..., F_k \in \mathcal{F}_1^{\uparrow}$ such that $\nu = l_{F_1} ... l_{F_k}$.*

Definition 4.1.2. *Let $C, C' \in \mathcal{F}_0^{\uparrow}$ be arbitrary chambers in $\mathscr{A}^{\uparrow}$, then the path ν from C to C' is* **minimal** *if it crosses once and only once each hyperplane separating C from C', and none of the other hyperplanes.*

The set of all positive minimal paths from C to C' is denoted by $(C \to C')$.

Lemma 4.1.3 (See [15], Lemma 11)**.** *All positive minimal paths from C to C' are homotopic in $\mathcal{S}^{\uparrow}$.*

Proof. Let us first consider a centered arrangement, so that there exists only one face, say G, of codimension n. Thus $\mu, \nu \in (C \to C')$ both lie in the boundary of the n-cell $[G, C]$ (remember the definition of a Salvetti complex in Section 2.4). Therefore, μ and ν are homotopic in the Salvetti complex, since they can be transformed into each other in the (contractible) n-cell $[G, C]$.

In the general case, let $H_1, ..., H_p$ be the hyperplanes separating C from C' and let $\mathscr{A}' = \{H_1, ..., H_p\}$ be the obtained arrangement. By this, we restrict our observation from the locally finite $\mathscr{A}^{\uparrow}$ to the finite arrangement $\mathscr{A}'$. The maximal codimension of intersections in $\mathscr{L}(\mathscr{A}')$ (see Definition 2.1.5) is $k \leq n$. Now let $F_1, ..., F_q \in \mathcal{F}_k^{\uparrow}$ be the faces of codimension k in $\mathscr{A}^{\uparrow}$ such that their support $\mathrm{supp}(F_j)$ lies in $\mathscr{L}(\mathscr{A}')$ and F_j lies in the same side as both C and C' with respect to all hyperplanes $H \in \mathscr{A}^{\uparrow} \backslash \mathscr{A}'$.

All positive minimal paths from C to C' are contained in the union of the k-cells $[F_i, (C)_{F_i}]$, otherwise a path would have to cross a hyperplane twice. This union $\cup_{i=1}^{q} [F_i, (C)_{F_i}]$ is contractible (see [15, p.613]), thus the positive minimal paths are homotopic. □

4.1.2 Generators

Fix a chamber $C_0 \in \mathcal{F}_0^{\uparrow}$ and remember the Definition 2.3.3 of C_F. Then we define for every $F \in \mathcal{F}_1^{\uparrow}$ the path

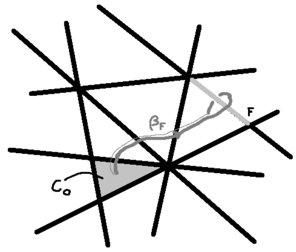
$$\beta_F := \mu_F l_F^2 \mu_F^{-1} \tag{4.1}$$

with fixed $\mu_F \in (C_0 \to (C_0)_F)$.

The homotopy class (relative to the base point) of β_F is denoted by $\overline{\beta_F}$.

Figure 4.1.4.

The path β_F.



Theorem 4.1.5 (Shown by Salvetti in [15]). *The fundamental group $\pi_1(\mathcal{S}^{\uparrow})$ is generated by the $\overline{\beta_F}$ with $F \in \mathcal{F}_1^{\uparrow}$.*

For a positive path $\nu = l_{F_1} ... l_{F_k}$ we define the loops

$$\beta_{F_i}^{\nu} := l_{F_1} ... l_{F_{i-1}} l_{F_i}^2 l_{F_{i-1}}^{-1} ... l_{F_1}^{-1} \text{ with } i \leq k. \tag{4.2}$$

From now on we use β_i as a shorthand for β_{F_i} and β_i^{ν} for $\beta_{F_i}^{\nu}$.

Furthermore, let $F_{i_1}, ..., F_{i_t}$ be the sequence obtained from $F_1, ..., F_k$ by recursively deleting faces F_i if the supporting hyperplane $\text{supp}(F_i)$ is the support of an odd number of faces of $F_i, ..., F_k$ (compare [15, p.614] or [1, p.22]) and define

$$\Sigma(\nu) := \{F_{i_1}, ..., F_{i_t}\}. \tag{4.3}$$

Thus, roughly speaking, $\Sigma(\nu)$ contains a face for every "loop" of ν around a hyperplane.

Remark 4.1.6. *Moreover, $\Sigma(\nu)$ is empty if and only if ν is minimal. In fact, $\Sigma(\nu)$ is empty implies ν crosses every hyperplane at most once, hence it is minimal. The converse is obvious.*

Lemma 4.1.7 (See [15], Lemma 12). *Let $\nu = l_{F_1} ... l_{F_k}$ be a positive path from C to C' and $\Sigma(\nu) = \{F_{i_1}, ..., F_{i_t}\}$, then*

$$\nu \simeq \beta_{i_t}^{\nu} ... \beta_{i_1}^{\nu} \mu,$$

where $\mu \in (C \to C')$.

Proof. We use induction on t. When t equals zero ν is minimal and, therefore it is homotopic to μ according to Lemma 4.1.3. Let t be greater than zero, then the path $l_{F_{i_t+1}} ... l_{F_k}$ is minimal by definition of $\Sigma(\nu)$. Furthermore, the hyperplane $H = \text{supp}(F_{i_t})$ is also crossed by $l_{F_{i_t+s}}$ for some $0 < s \leq k - i_t$.

Let C_{i-1} denote the start of l_{F_i} in ν and C_i the end of l_{F_i} in ν, $C = C_0$ and $C' = C_k$. For $\mu \in (C_{i_t-1} \to C_{i_t+s})$ the path $l_{F_{i_t}} \mu$ is a minimal path from C_{i_t} to C_{i_t+s}, note that $l_{F_{i_t}}$ means to go from C_{i_t} to C_{i_t-1} in positive direction here. Hence, the path $l_{F_{i_t}} \mu$ is homotopic to $l_{F_{i_t+1}} ... l_{F_{i_t+s}}$, which is also minimal. Therefore, we have

$$\begin{aligned}
\nu = l_{F_1} ... l_{F_k} &\simeq l_{F_1} ... l_{F_{i_t}} (l_{F_{i_t}} \mu) l_{F_{i_t+s+1}} ... l_{F_k} \\
&= (l_{F_1} ... l_{F_{i_t-1}}) l_{F_{i_t}}^2 (l_{F_{i_t-1}}^{-1} ... l_{F_1}^{-1}) (l_{F_1} ... l_{F_{i_t-1}}) \mu (l_{F_{i_t+s+1}} ... l_{F_k}) \\
&= \beta_{i_t}^{\nu} (l_{F_1} ... l_{F_{i_t-1}}) \mu (l_{F_{i_t+s+1}} ... l_{F_k}).
\end{aligned}$$

For the path $\nu' = (l_{F_1} \dots l_{F_{i_t-1}})\mu(l_{F_{i_t+s+1}} \dots l_{F_k})$ we know that $\mu(l_{F_{i_t+s+1}} \dots l_{F_k})$ is minimal. Moreover, ν' crosses every hyperplanes that ν crosses except for $H = \text{supp}(F_{i_t})$, which is crossed two times less by ν'. So $\Sigma(\nu') = \{F_{i_1}, \dots, F_{i_t-1}\}$ and thus the assumption holds for ν' by induction. □

Lemma 4.1.8 (See [15], Corollary 12)**.** *Let $F, G \in \mathcal{F}_1^{\uparrow}$ be two faces with the same support. Moreover, let $\nu = l_{F_1} \dots l_{F_k}$ be a positive path from C_0 to $(C_0)_G$ and $F_{i_1}, \dots, F_{i_t}$ be the faces whose support does not seperate C_0 from $(C_0)_F$. Then*

$$\beta_F \simeq \left(\prod_{j=t}^{1} \beta_{i_j}^{\nu} \right)^{-1} \beta_G \left(\prod_{j=t}^{1} \beta_{i_j}^{\nu} \right).$$

Proof. For $1 \leq i \leq k$, let C_{i-1}, C_i denote the first and the second end of l_{F_i} in ν, thus $(C_0)_G = C_k$ and ν starts in C_0 as required. Furthermore, let C_{k+1} denote the chamber on the other side of G, hence $C_{k+1} = \text{op}((C_0)_G, G)$. Set $C' = \text{op}((C_0)_F, F)$ and let $\mu \in (C_{k+1} \to C')$, then we have the positive path $\nu l_G \mu$. Since νl_G is minimal and $\nu l_G \mu$ crosses a hyperplane twice if and only if the support of F_i does not seperate C_0 from $(C_0)_F$, the path μ has to cross this hyperplane again and thus $\Sigma(\nu l_G \mu) = \{F_{i_1}, \dots, F_{i_t}\}$. Moreover, we know for the positive path $\nu l_G \mu l_F$ that $\Sigma(\nu l_G \mu l_F) = \{F_{i_1}, \dots, F_{i_t}, G\}$.

Thus we deduce by Lemma 4.1.7:

$$\nu l_G \mu \simeq \left(\prod_{j=t}^{1} \beta_{i_j}^{\nu} \right) \varepsilon l_F,$$

$$\nu l_G \mu l_F \simeq \beta_G \left(\prod_{j=t}^{1} \beta_{i_j}^{\nu} \right) \varepsilon,$$

where $\varepsilon \in (C_0 \to (C_0)_F)$.

Multiplying with l_F yields:

$$\left(\prod_{j=t}^{1} \beta_{i_j}^{\nu} \right) \varepsilon l_F^2 \simeq \beta_G \left(\prod_{j=t}^{1} \beta_{i_j}^{\nu} \right) \varepsilon.$$

By considering that $\beta_F = \varepsilon l_F^2 \varepsilon^{-1}$, we get:

$$\beta_F = \varepsilon l_F^2 \varepsilon^{-1} \simeq \left(\prod_{j=t}^{1} \beta_{i_j}^{\nu} \right)^{-1} \beta_G \left(\prod_{j=t}^{1} \beta_{i_j}^{\nu} \right).$$

□

In particular, Lemma 4.1.8 means that it is not so important where a path makes a loop around a hyperplane.

4.1.3 Relations

Let G be a codimension one face of $\mathscr{A}^{\uparrow}$, and $C \in \mathcal{F}_0^{\uparrow}$ be a chamber containing it. Denote C' to be the opposite chamber of C with respect to G and let $\nu = l_{F_1} ... l_{F_k}$ be a positive minimal path from C to C'. Then define the subset $h(G) := \{F_1, ..., F_k\}$ of $\mathcal{F}_1^{\uparrow}$. The ordering of $h(G)$ is well-defined up to cyclic permutation.

Furthermore, let $F_{i+k} \in \mathcal{F}_1^{\uparrow}$ denote the face which is also contained in $\operatorname{supp} F_i$, only separated from F_i by G. In particular, the face G is also contained in F_{i+k}. Define the path

$$\alpha_G(C) = l_1 ... l_{2k}, \tag{4.4}$$

which corresponds to a circle around G.

Salvetti introduced a set of relations associated with G in [15, p.613]:

$$R_G : \quad \beta_k ... \beta_1 \simeq \beta_{\sigma(k)} ... \beta_{\sigma(1)},$$

where σ is a cyclic permutation of $(1, ..., k)$. Moreover, for $\nu \in (C \to C')$ we have

$$\beta_{\sigma(k)} ... \beta_{\sigma(1)} \simeq \nu \alpha_G((C_0)_G) \nu^{-1}.$$

The fundamental group of $\mathcal{M}(\mathscr{A}^{\uparrow})$ can be presented as generated by the $\overline{\beta_F}$ together with the relations R_G. This is one of the results Salvetti showed in his paper [15, p.616]. Thus:

Theorem 4.1.9. *Let $\mathscr{A}$ be a complexified essential toric arrangement as above, then the fundamental group of the complement of the affine arrangement $\mathscr{A}^{\uparrow}$ is presented as*

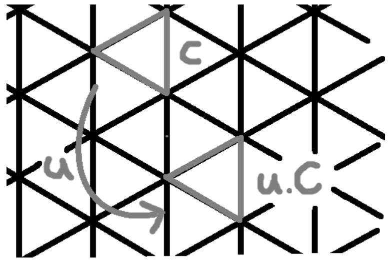
$$\pi_1(\mathcal{S}^{\uparrow}) = \langle \overline{\beta_F}, F \in \mathcal{F}_1^{\uparrow} | R_G, G \in \mathcal{F}_2^{\uparrow} \rangle.$$

4.2 Connection to $\mathscr{A}$

Let $\mathscr{A}$ and $\mathscr{A}^{\uparrow}$ be arrangements as above, C_0 is again a fixed chamber in $\mathcal{F}_0^{\uparrow}$ and choose a basis of $u_1, ..., u_n$ of the character lattice Λ.

The set of characters Λ acts on V ($\mathbb{C}^n$ or $\mathbb{R}^n$) by translation, for $u \in \Lambda$ and $z \in \mathbb{C}^n$ we denote the translation of z by u with $u.z$ (compare Sections 1.5 and 1.3.2). Furthermore, if ν is a path on $\mathcal{G}^\uparrow$ we write $u.\nu$ for the path obtained by translating ν by u. This is valid action of Λ on $\mathcal{G}^\uparrow$, since $\mathscr{A}^\uparrow$ is a periodic affine arrangement corresponding to Λ and therefore, the path $u.\nu$ lies again on $\mathcal{G}^\uparrow$.

Figure 4.2.1. **The action of Λ on $\mathcal{F}^\uparrow$ by translation.**



Now fix a generic point x_0 in C_0, such that the straight line segment s_i from x_0 to $u_i.x_0$ meets only faces of codimension at most one in $\mathscr{A}^\uparrow$ for all $i = 1, ..., n$.

Definition 4.2.2. *Let $u_1, ..., u_n$ be a basis of Λ as above, then ω_i denotes the path from C_0 to $u_i.C_0$ obtained by crossing the faces met by s_i. Moreover, define recursively the paths $\omega_i^{(k)} := \omega(u_i.\omega_i^{(k-1)})$ for a non-negative k. Similarly, define $\omega_i^{(-1)} := u_i^{-1}.\omega_i^{-1}$ and $\omega_i^{(-k)} = \omega_i^{(-1)}(u_i^{-1}.\omega_i^{(1-k)})$ for negative exponents. For an arbitrary $u \in \Lambda$ with presentation $u = u_1^{q_1} ... u_n^{q_n}$ define*

$$\omega_u := \omega_1^{(q_1)} u_1^{q_1}.\omega_2^{(q_2)} ... \left(\prod_{j=1}^{n-1} u_j^{q_j}\right).\omega_n^{(q_n)}.$$

Furthermore, define the paths

$$\tau_i := p(\omega_i), \tau_u := p(\omega_u)$$

in $\mathcal{M}(\mathscr{A})$.

Note that, in general, the paths ω_u are neither minimal nor positive. In fact, ω_u is positive if and only if all exponents q_i are non-negative. Moreover, for an arbitrary i we decude that $\omega_i^{(k)}$ is minimal if and only if k non-negative, thus in this case $\omega_i^{(k)}$ is a positive minimal path.

Lemma 4.2.3 (See Lemma 5.8 in [1])**.** *In $\mathcal{M}(\mathscr{A})$, the path $p(\omega_i^{(k)})$ equals τ_i^k and $\tau_i\tau_j$ is homotopic to $\tau_j\tau_i$ for all i, j. Let $\varepsilon : \pi_1(\mathcal{M}(\mathscr{A})) \to \pi_1(T_\Lambda)$ be induced by the inclusion $\mathcal{M}(\mathscr{A}) \to T_\Lambda$, then $\pi_1(T_\Lambda)$ is generated by the $\varepsilon\overline{\tau_i}$, where $\overline{\tau_i}$ denotes the homotopy class of τ_i.*

We skip the proof and consider the action of Λ on $\mathcal{F}^{\uparrow}$ by translation. As above, it is a valid action. Now we construct a fundamental domain for this action.

Definition 4.2.4. *Consider the Minkowski sum $X' := s_1 + ... + s_n \subset \mathbb{R}^n$. Let Q denote the set of all faces $F \in \mathcal{F}^{\uparrow}$ which have a non-empty intersection with X'. Moreover, set $Q_i := Q \cap \mathcal{F}_i^{\uparrow}$. In particular, the set Q contains C_0 and the faces crossed by some s_i are contained in Q_1.*

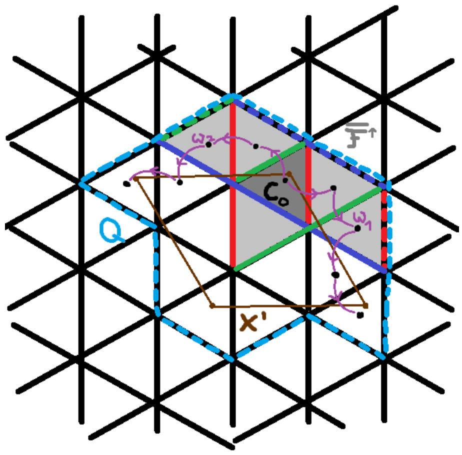
The polytope X' is a cell complex, so let $\mathcal{F}(X')$ denote the set of its faces. For the next definition we need to define another set, namely the set $\mathcal{B}$ of all faces of X' which intersect the convex hull of $\{s_i \backslash \{u_i.x_0\} | i = 1, ..., n\}$. Notice that all faces of X' are translations of faces in $\mathcal{B}$ by a combination of basis elements $u_1^{m_1} ... u_n^{m_n}$ with $m_i \in \{0, 1\}$. Let D be the union of the faces in $\mathcal{B}$, then D is a fundamental domain of the action of Λ on $\mathbb{R}^n$.

Definition 4.2.5. *Define the set*

$$\begin{aligned}\overline{\mathcal{F}^{\uparrow}} :=& \{F \in Q | F \cap B = \emptyset \text{ for all } B \in \mathcal{F}(X') \backslash \mathcal{B}\} \\ =& \{F \in Q | F \cap X' \backslash D = \emptyset\}.\end{aligned}$$

Indeed, a face B of X' is not in $\mathcal{B}$ if and only if B is contained in $X' \backslash D$. Particularly, for all faces in $\overline{\mathcal{F}^{\uparrow}}$ it holds that they do not intersect the line segments $\prod_{j \neq i} u_j^{m_j}.s_i$ for all i and all $m_j \in \{0, 1\}$ with at least one $m_j \neq 0$.

Figure 4.2.6. **The set $\overline{\mathcal{F}^{\uparrow}}$ in $\mathscr{A}_2^{\uparrow}$.**



Remark 4.2.7. *The set $\overline{\mathcal{F}^{\uparrow}}$ always contains at least one representative for each orbit of the action of Λ on $\mathcal{F}^{\uparrow}$. If it is constructed in a particular way (with respect to the*

choice of C_0, x_0 and the basis of Λ), it contains exactly one representative. Thus it is a fundamental domain for the action of Λ on $\mathcal{F}^\uparrow$ in this case (see Chapter 5).

From now on we assume that C_0, x_0 and the basis of Λ are chosen in a way, such that $\overline{\mathcal{F}^\uparrow}$ is a fundamental domain.

Definition 4.2.8. *For a face $F \in \mathcal{F}^\uparrow$ let $\overline{F}$ be the unique element in $\Lambda F \cap \overline{\mathcal{F}^\uparrow}$ and the translation is obtained by $u_F \in \Lambda$, i.e., the face F is equal to the translation $u_F.\overline{F}$. Furthermore, define the path*

$$\Gamma_F := \omega_{u_F}(u_F.\beta_{\overline{F}})\omega_{u_F}^{-1}$$

for all $F \in \mathcal{F}_1^\uparrow$.

Remark 4.2.9 (See Remark 5.12 in [1])**.**

(1) For all $F \in \overline{\mathcal{F}^\uparrow}$ and arbitrary $u \in \Lambda$ holds $p(\Gamma_{uF}) = \tau_u p(\Gamma_F)\tau_u^{-1}$.

(2) If $F \in \overline{\mathcal{F}^\uparrow}_1$, then $\Gamma_F = \beta_F$.

(3) If $F \in Q$ and $u_1, ..., u_n$ is the basis of Λ, then $u_F = \prod_{i=1}^n u_i^{a_i}$ with $a_i \geq 0$ for all i.

(4) Since X' is convex, the set Q_0 contains the vertices of a positive minimal path between two elements of Q_0.

Definition 4.2.10. *Define $\Omega_i := \{F \in \mathcal{F}_1^\uparrow | F$ is crossed by $\omega_i^{(k)}$ for some $k\}$ for all $i = 1, ..., n$ and $\Omega := \bigcup_{i=1}^n \Omega_i$.*

Lemma 4.2.11 (See Lemma 5.14 in [1])**.** *The subgroup of $\pi_1(\mathcal{M}(\mathscr{A}^\uparrow))$ generated by all $\overline{\beta_F}$ with $F \in \Omega_i$ is contained in the subgroup generated by all $\overline{\Gamma_F}$ with $F \in \Omega_i$ for all $i = 1, ..., n$.*

Our proof corrects the proof given in [1].

Proof. Let without loss of generality F be in Ω_1 with $F = u_1^k\overline{F}$. Let us first consider the case k is non-negative, then $\omega_{u_1^k} = \omega_i^{(k)}$ is positive minimal (recall the discussion after Definition 4.2.2). If $(C_0)_F = (u_F.C_0)_F$ then $\beta_F \simeq \Gamma_F$ holds by construction.

So let k be greater than zero and $(C_0)_F \neq (u_F.C_0)_F$ and $m(F)$ denote the length of a minimal path from $u_F.C_0$ to $(u_F.C_0)_F$, i.e., the number of hyperplanes both ω_{u_F} and $u_F.\beta_{\overline{F}}$ cross. For $m(F) = 0$ the path β_F is again homotopic to Γ_F by construction.

Suppose $m(F)$ is positive and let F_i for $i = 1, ..., m = m(F)$ be the faces which ω_{u_F} crosses after $F = F_0$ in the translation of $\overline{\mathcal{F}^\uparrow}$ by u_F, thus

$$\omega_{u_F} = \mu l_{F_0} ... l_{F_m}$$

where $\mu \in (C_0 \to (C_0)_F)$. Moreover, we obtain

$$u_F.\beta_{\overline{F}} \simeq l_{F_m} ... l_{F_1} l_{F_0}^2 l_{F_1}^{-1} ... l_{F_m}^{-1}$$

and $\mu_i \simeq \mu l_{F_0} ... l_{F_{i-1}}$ for $\mu_i \in (C_0 \to (C_0)_{F_i})$. So for Γ_F we deduce

$$\begin{aligned}
\Gamma_F &\overset{def}{=} \omega_{u_F}(u_F.\beta_{\overline{F}})\omega_{u_F}^{-1} \\
&\simeq (\mu l_{F_0} ... l_{F_{m-1}}) l_{F_m} (l_{F_m} ... l_{F_1} l_{F_0}^2 l_{F_1}^{-1} ... l_{F_m}^{-1}) l_{F_m} (\mu l_{F_0} ... l_{F_{m-1}})^{-1} \\
&\simeq \mu_m l_{F_m}^2 \mu_m^{-1} (\mu l_{F_0} ... l_{F_{m-1}}) l_{F_{m-1}} ... l_{F_1} l_{F_0}^2 l_{F_1}^{-1} ... l_{F_{m-1}}^{-1} (\mu l_{F_0} ... l_{F_{m-1}})^{-1} \mu_m l_{F_m}^{-2} \mu_m^{-1} \\
&\simeq \beta_m ... \beta_1 \beta_0 \beta_1^{-1} ... \beta_m^{-1}
\end{aligned}$$

where the last step uses $\beta_i \simeq \mu_i l_{F_i}^2 \mu_i^{-1}$. Therefore,

$$\beta_F = \beta_0 \simeq (\beta_m ... \beta_1)^{-1} \Gamma_F \, \beta_m ... \beta_1$$

where for the β_i with $i \geq 1$ holds that $m(F_i) < m(F)$. By induction, the path β_F is homotopic to a product of the Γ_G with $G \in \Omega_1$.

Assume k is negative and let $\nu \in ((C_0)_F \to C_0)$ be the path that follows the segments s_1, then we argue by induction on the length $\mathrm{d}(F)$ of ν. The induction starts at $\mathrm{d}_0 = \min\{\mathrm{d}(F) | u_F = u_1^{-1}\} \geq 0$ depending on the choice of $\overline{\mathcal{F}^\uparrow}$, particularly $k = -1$ and $(C_0)_F \neq (u_F.C_0)_F$, and in this case Γ_F is homotopic to β_F.

Assume $\mathrm{d}(F)$ is positive. Now we have to consider the two cases, $(C_0)_F \neq (u_F.C_0)_F$ and $(C_0)_F = (u_F.C_0)_F$. So suppose $(C_0)_F \neq (u_F.C_0)_F$. Let μ be a minimal positive path from C_0 to $(C_0)_F$ following s_1, then we know

$$\Gamma_F \simeq \nu^{-1} l_F^2 \nu \text{ and } \beta_F = \mu l_F^2 \mu^{-1}.$$

Thus

$$\beta_F = (\mu\nu)\nu^{-1} l_F^2 \nu (\mu\nu)^{-1} \simeq (\mu\nu)\Gamma_F(\mu\nu)^{-1}.$$

The path $\mu\nu$ is positive. Now $\Sigma(\mu\nu)$ be the set of all faces $F \in \mathcal{F}_1^\uparrow$ crossed by μ. Therefore, by Lemma 4.1.7 the path $\mu\nu$ is homotopic to the product $\beta_{\mathrm{d}(F)}^\mu ... \beta_1^\mu$. So for

β_F we conclude

$$\beta_F \simeq \left(\prod_{\Sigma(\mu\nu)} \beta_i^\mu \right) \Gamma_F \left(\prod_{\Sigma(\mu\nu)} \beta_i^\mu \right)^{-1}. \tag{4.5}$$

Since the faces F' in $\Sigma(\mu\nu)$ are crossed by μ, i.e., also by s_1, they lie in Ω_1. Furthermore, the value $\mathrm{d}(F')$ is lower than $\mathrm{d}(F)$ for all F' in $\Sigma(\mu\nu)$.

Now let us consider a face F with $F = u_1^k.\overline{F}$, where k is negative, and $(C_0)_F = (u_F.C_0)_F$. Set $\rho \in (C_0 \to u_F.C_0)$ and $\xi \in (u_F.C_0 \to C_0)$, the paths which follow the segment s_1, then

$$\beta_F \simeq \rho\, u_F.\beta_{\overline{F}}\, \rho^{-1} \text{ and } \Gamma_F \simeq \xi^{-1} u_F.\beta_{\overline{F}}\, \xi.$$

Thus

$$\beta_F \simeq (\rho\xi)\, \Gamma_F\, (\rho\xi)^{-1} \simeq \left(\prod_{\Sigma(\rho\xi)} \beta_i^\rho \right) \Gamma_F \left(\prod_{\Sigma(\rho\xi)} \beta_i^\rho \right)^{-1} \tag{4.6}$$

by Lemma 4.1.7. As above, the value $\mathrm{d}(F')$ is lower than $\mathrm{d}(F)$ for all F' in $\Sigma(\rho\xi)$. By induction, it follows that the β_i^μ in (4.5) and the β_i^ρ in (4.6) are homotopic to a product of Γ_F with $F \in \Omega_i$. Hence, the claim follows. □

Lemma 4.2.12 (See Lemma 5.15 in [1]). *The fundamental group $\pi_1(\mathcal{M}(\mathscr{A}^\uparrow))$ is generated by all $\overline{\Gamma_F}$ with $F \in \Omega$.*

Proof. For $F \in \mathcal{F}_1^\uparrow$ we have to show that $\overline{\beta_F}$ is generated by the $\overline{\Gamma_F}$ with $F \in \Omega$, because we then deduce the claim from Theorem 4.1.9. The support H of F is crossed by $\omega_i^{(k)}$ for some $i \in \{1, ..., n\}$ and some $k \in \mathbb{Z}$ ("every hyperplane is cut by the coordinate axes"). Let G denote the face in $\mathcal{F}_1^\uparrow$ where $\omega_i^{(k)}$ crosses H. By Lemma 4.1.8, we get that the path β_F is homotopic to the product of β_G and other $\beta_{G'}$ with $G' \in \Omega$. Thus the homotopy class of this product is generated by $\overline{\Gamma_{F'}}$ with $F' \in \Omega$ by Lemma 4.2.11. □

4.3 On the torus

Now we have to transfer the above results to the toric arrangement $\mathscr{A}$.

Definition 4.3.1. *For $F \in \mathcal{F}_1^\uparrow$ let γ_F be the path obtained as the image of Γ_F under the projection p, i.e., the path $\gamma_F := p(\Gamma_F)$.*

Definition 4.3.2. *Let F be in Q_1 and $\mu \in (u_F.C_0 \to (u_F.C_0)_F)$, then define the path*

$$\Delta_F := \prod_{F' \in \Sigma(w_{u_F}\mu)} \beta_{F'}.$$

In particular, the faces $F' \in \Sigma(w_{u_F}\mu)$ lie in Q by choice of F. Furthermore, define the paths in $\mathcal{M}(\mathscr{A})$ as follows

$$\delta_F := p(\Delta_F) \quad \textit{and} \quad \gamma_F^\delta := \delta_F^{-1}\gamma_F\delta_F.$$

Note that our definition of Δ_F is different from the one given in [1]. In order to account for the correction of the proof of Lemma 4.2.11, we had to change it to this defintion.

We define a toric equivalent to the relations R_G in $\mathcal{M}(\mathscr{A}^\uparrow)$ introduced by Salvetti in [15]. So recall for $G \in \mathcal{F}_2^\uparrow$ the relations in $\mathcal{M}(\mathscr{A}^\uparrow)$:

$$R_G: \quad \beta_k ... \beta_1 \simeq \beta_{\sigma(k)} ... \beta_{\sigma(1)}, \tag{4.7}$$

where $h(G) = \{F_1, ..., F_k\}$ (see Section 4.1.3 for the definition) and σ is a cyclic permutation of $\{1, ..., k\}$.

For $G \in \overline{\mathcal{F}^\uparrow}_2$, we define

$$R_G^\downarrow: \quad \gamma_k^\delta ... \gamma_1^\delta \simeq \gamma_{\sigma(k)}^\delta ... \gamma_{\sigma(1)}^\delta, \tag{4.8}$$

where again $h(G) = \{F_1, ..., F_k\}$ and σ is a cyclic permutation of $\{1, ..., k\}$. As above, the path γ_i^δ denotes $\gamma_{F_i}^\delta$ for F_i.

Lemma 4.3.3. *If F is a face in Q_1, then γ_F^δ is homotopic to $p(\beta_F)$.*

Proof. By Lemma 5.16 in [1], we have $\beta_F \simeq \Delta_F^{-1}\Gamma_F\Delta_F$ for $F \in Q_1$. Thus by the use of the projection p the claim follows immediately. □

By Theorem 4.2.12, the path Δ_F is homotopic to a product of the Γ_F with $F \in \Omega$. Let $M(F)$ be the ordered set of faces in Ω, such that

$$\Delta_F \simeq \prod_{F' \in M(F)} \Gamma_{F'}.$$

Lemma 4.3.4 (Rectifies Lemma 5.20 in [1])**.** *For $F \in Q_1$ and $M(F) \subset \Omega$ as above, we have*

$$\delta_F \simeq \prod_{F' \in M(F)} \tau_{u_{F'}} \gamma_{\overline{F'}} \tau_{u_{F'}}^{-1}.$$

In particular, the path γ_F^δ can be written as a word in the $\tau_1, ..., \tau_n$ and γ_F with $F \in \overline{\mathcal{F}^\uparrow}_1$.

Proof. This is an easy computation using Remark 4.2.9. □

Lemma 4.3.5 (See Lemma 5.2 in [1]). *$\pi_1(\mathcal{M}(\mathscr{A}))$ is the semidirect product of $\pi_1(\mathcal{S}^{\uparrow})$ and $\pi_1(T_\Lambda)$.*

Immediately we get a presentation of the fundamental group of $\mathcal{M}(\mathscr{A})$ with the preceding lemmata. According to Lemma 4.2.3, we know that the $\overline{\tau_i}$ generate the fundamental group of the torus T_Λ and the fundamental group of $\mathcal{M}(\mathscr{A}^{\uparrow})$ is generated by the $\overline{\Gamma_F}$ by Theorem 4.2.12. Furthermore, we have $\gamma_F^\delta \simeq p(\beta_F)$ for $F \in Q_1$ by Lemma 4.3.3 and γ_F^δ can be written as a word in the τ_i and γ_F with $F \in \overline{\mathcal{F}^{\uparrow}}_1$ by Lemma 4.3.4. Thus the relations $R_G^{\downarrow}$ can be expressed in terms of the τ_i and the γ_F with $F \in \overline{\mathcal{F}^{\uparrow}}_1$. So we get:

Theorem 4.3.6 (See Theorem 5.22 in [1]). *The fundamental group $\pi_1(\mathcal{M}(\mathscr{A}))$ of the toric arrangement $\mathscr{A}$ is presented as*

$$\langle \overline{\tau_1}, ..., \overline{\tau_n}; \overline{\gamma_F}, F \in \Omega \cap \overline{\mathcal{F}^{\uparrow}}_1 | \tau_i\tau_j \simeq \tau_j\tau_i \textit{ for } i,j = 1, ..., n; R_G^{\downarrow}, G \in \overline{\mathcal{F}^{\uparrow}}_2 \rangle.$$

5 Fundamental Domain

As stated after Defintion 4.2.5 of $\overline{\mathcal{F}^{\uparrow}}$, the set $\overline{\mathcal{F}^{\uparrow}}$ is in general not a fundamental domain of the action of Λ on $\mathcal{F}^{\uparrow}$. In this chapter we discuss in which cases $\overline{\mathcal{F}^{\uparrow}}$ contains more than one representative of some orbit and how the chamber C_0 in $\mathscr{A}^{\uparrow}$, the point x_0 and the basis elements $u_1, ..., u_n$ of Λ have to be chosen in order for $\overline{\mathcal{F}^{\uparrow}}$ to contain exactly one representative for each orbit.

5.1 Problems

Recall the general settings. The set $\mathscr{A}$ is a (complexified and essential) toric arrangement and the affine arrangement $\mathscr{A}^{\uparrow}$ is its lifting as in Section 3.2. Their face posets are denoted by $\mathcal{F}$ respectively $\mathcal{F}^{\uparrow}$ while $\mathcal{F}_i$ respectively $\mathcal{F}_i^{\uparrow}$ are their subsets of faces of codimension i. The characters $u_1, ..., u_n$ form a basis of the lattice Λ, $C_0 \in \mathcal{F}_0^{\uparrow}$ is a fixed chamber and x_0 is a generic point in C_0, such that the straight line segment s_i from x_0 to $u_i.x_0$ only meets faces of codimension at most one in $\mathscr{A}^{\uparrow}$ for all $i = 1, ..., n$.

We want to get a suitable fundamental domain for the action of the character lattice Λ on the set $\mathcal{F}^{\uparrow}$ of faces of $\mathscr{A}^{\uparrow}$, such that we are able to reduce our consideration about $\pi_1(\mathcal{M}(\mathscr{A}))$ to the paths corresponding to the faces in this fundamental domain.

Let us recall the construction of $\overline{\mathcal{F}^{\uparrow}}$ from Section 4.2 (see also Figure 4.2.6). We start with the Minkowski sum $X' = s_1 + ... + s_n \subset \mathbb{R}^n$. We then define the set $Q = \{F \in \mathcal{F}^{\uparrow} | F \cap X' \neq \emptyset\}$ and let $\mathcal{B}$ be the set of all faces of the polyhedron X' which intersect the convex hull of $\{s_i \backslash \{u_i.x_0\} | i = 1, ..., n\}$. We get the subset $D = \bigcup_{F \in \mathcal{B}} F$ of X' as a fundamental domain of the action of Λ on $\mathbb{R}^n$.

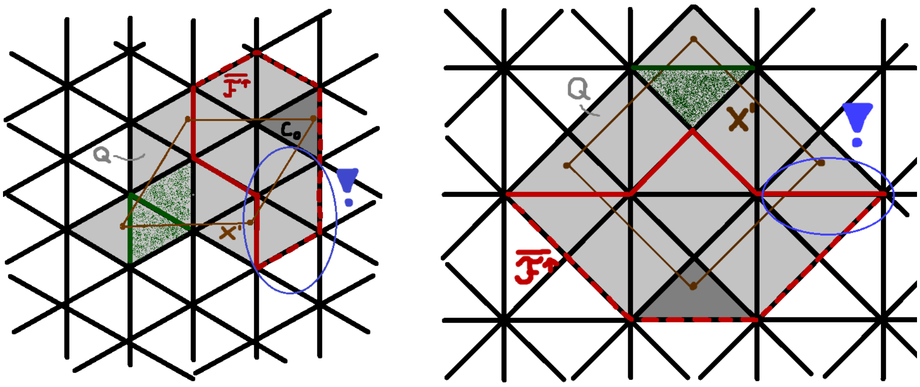
Now we define

$$\begin{aligned}\overline{\mathcal{F}^{\uparrow}} :=& \{F \in Q | F \cap B = \emptyset \text{ for all } B \in \mathcal{F}(X') \backslash \mathcal{B}\} \\ =& \{F \in Q | F \cap X' \backslash D = \emptyset\}.\end{aligned}$$

In this way, the authors d'Antonio and Delucchi try in [1] to transfer the property of D to be a fundamental domain of the action of Λ on $\mathbb{R}^n$ to $\overline{\mathcal{F}^{\uparrow}}$ and the action of Λ

on $\mathcal{F}^{\uparrow}$. Unfortunately this does not always work, as sometimes we get more than one representative for some orbits (see Figure 5.1.1).

Figure 5.1.1. **Counterexamples.**



Two valid constructions of the set $\overline{\mathcal{F}^{\uparrow}}$, in both cases $\overline{\mathcal{F}^{\uparrow}}$ is not the wished set of representatives.

Remark 5.1.2. *The set $\overline{\mathcal{F}^{\uparrow}}$ is not in general a fundamental domain of the action of Λ on $\mathcal{F}^{\uparrow}$.*

Proof. As counterexample take a look at Figure 5.1.1, where we have the Weyl arrangement corresponding to the Weyl group A_2 with a valid choice of C_0, x_0 and the basis of Λ. In the case when $u_2.s_1$ and $u_1.s_2$ cross the same face F ($\neq u_1u_2.C_0$) in $\mathcal{F}^{\uparrow}$, we get that $u_1^{-1}.F$ as well as $u_2^{-1}.F$ are in $\overline{\mathcal{F}^{\uparrow}}$ and thus $\overline{\mathcal{F}^{\uparrow}}$ contains two representatives of the corresponding orbit, one met by s_1 and the other by s_2. □

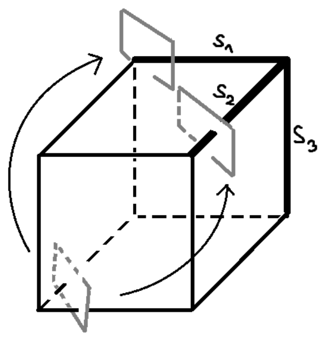
Corollary 5.1.3. *Let $\mathscr{A}$ be a toric arrangement on the 2-dimensional torus with u_1, u_2 as the basis of Λ. Then $\overline{\mathcal{F}^{\uparrow}}$ contains two representatives of some orbit of the action of Λ on $\mathcal{F}^{\uparrow}$ if and only if there exists a face $F(\neq u_1u_2.C_0)$ in this orbit which is met by both $u_2.s_1$ and $u_1.s_2$.*

Indeed, the set $\overline{\mathcal{F}^{\uparrow}}$ contains by construction (at least) one representative for each orbit (Λ acting on $\mathcal{F}^{\uparrow}$). Since D is a fundamental domain, it intersects each orbit. Define

$$\hat{u}_i := u_1 \ldots u_{i-1} u_{i+1} \ldots u_n, \tag{5.1}$$

then the appearance of more than one representative is connected with the constellation of the segments $\hat{u}_i.s_i$ as seen in Figure 5.1.1 or in Figure 5.1.4 below. In fact, it depends on the way the spans of the segments $\hat{u}_i.s_i$ meet the faces in $\mathcal{F}^{\uparrow}$ of $\mathscr{A}^{\uparrow}$. Note that the spans of the segments $\hat{u}_i.s_i$ are exactly the faces in $X' \backslash D$.

Figure 5.1.4. **Constellation of the segments s_i.**



For a face F in Q, let $\mathcal{N}(F)$ denote the set of faces in $X'\backslash D$ which have a non-empty intersection with F. Then $\mathcal{N}(F)$ consists of flags of faces (of X'). Let $N_1^F, ..., N_k^F$ denote the smallest faces of this flags, whereat N_i^F is not equal to N_j^F for different i and j, and k an integer from zero to n.

Lemma 5.1.5. *The set $\overline{\mathcal{F}^\uparrow}$ contains more than one representative of some orbit of the action of Λ on $\mathcal{F}^\uparrow$ if and only if there exists a face F in this orbit, such that k is greater than one and F is not met by the affine subspace generated by the common facet of $N_1^F, ..., N_k^F$, which is $\bigcap_{i=1}^k N_i^F$.*

Proof. Let say $F' \neq F''$ are two representatives of the same orbit in $\overline{\mathcal{F}^\uparrow}$. If D meets an orbit of the action of Λ on $\mathcal{F}^\uparrow$ twice then the representatives have to be met by the boundary of D. Hence, both F' and F'' are met by a face in D of lower dimension than n. Say $G' := \text{span}\{s_i | i \in I\} \subsetneq D$ is a face of lowest dimension which intersects F' and $G'' := \text{span}\{s_j | j \in J\}$ is a face of lowest dimension which intersects F''.

Since $F' \neq F''$ it follows that $I \neq J$. Moreover, since F', F'' lie in the same orbit neither G' is a subset of G'' nor G'' of G' and there exists some $u \in \Lambda$ such that $F' = u.F''$. More precisely, it holds $u = \prod_{i \in I} u_i (\prod_{j \in J} u_j)^{-1}$. So set

$$F = \hat{u}_I.F' = \hat{u}_J.F'',$$

where $\hat{u}_I := \prod_{i \notin I} u_i$ and $\hat{u}_J := \prod_{j \notin J} u_j$.

The face F intersects $X'\backslash D$. In particular, it meets the faces $\hat{u}_I.G' \neq \hat{u}_J.G''$ which are parallel to G' respectively G'' and lie in $\mathcal{N}(F)$. Furthermore, the faces $\hat{u}_I.G'$ and $\hat{u}_J.G''$ are the smallest elements of some flags in $\mathcal{N}(F)$ by the choice of G' respectively G''.

Moreover, the fact that F intersects the affine subspace generated by those faces contradicts the assumption, that F' and F'' lie in $\overline{\mathcal{F}^\uparrow}$.

On the other hand, let F be a face, such that $\mathcal{N}(F)$ contains more than one "smallest" element and F does not intersect the span of their intersection. Thus there exist some sets $J_i \subsetneq \{1, ..., n\}$ such that N_i^F is the translation by $\hat{u}_{J_i}$ of the face $G_i := \text{span}\{s_j | j \in J_i\}$ in D for all $i \leq k$.

The faces $\hat{u}_{J_1}^{-1}.F \neq \hat{u}_{J_k}^{-1}.F$ (of $\mathscr{A}^\uparrow$) are met by the faces G_1 respectively G_k (of X') in D and lie in $\overline{\mathcal{F}^\uparrow}$, since F does not intersect the span of the intersection of the N_i^F. Thus $\overline{\mathcal{F}^\uparrow}$ contains more than one representative of the orbit of F. $\square$

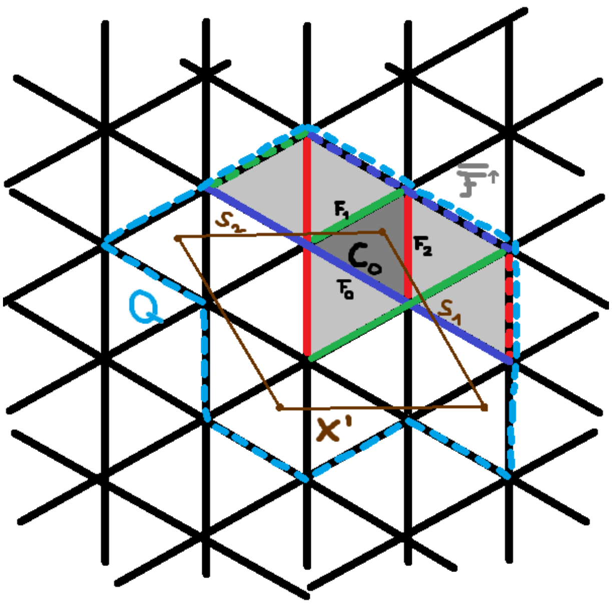
So we have to choose C_0, x_0 and the basis elements $u_1, ..., u_n$ in such a way, that those "double crosses" do not occur.

5.2 Choice for $\mathscr{A}_2^\uparrow$

Let us start with the Weyl group A_2 of rank 2 to understand the construction and then generalize the idea.

Lemma 5.2.1. *Let $\mathscr{A}_2$ be the toric Weyl arrangement corresponding to the Weyl group A_2. Then there exist a chamber $C_0 \in \mathcal{F}_0^\uparrow$, a point x_0 in C_0 and a basis u_1, u_2 of the coroot lattice Λ, such that the set $\overline{\mathcal{F}^\uparrow}$ is a fundamental domain of the action of Λ on $\mathcal{F}^\uparrow(\mathscr{A}_2)$.*

Figure 5.2.2. **Construction of $\overline{\mathcal{F}^\uparrow}$ for $\mathscr{A}_2^\uparrow$.**



Proof. Choose an arbitrary chamber $C_0 \in \mathcal{F}_0^{\uparrow}$ of $\mathscr{A}_2^{\uparrow}$, which is a 2-simplex. Then let the hyperplanes H_1 and H_2 denote the walls of C_0 corresponding to the simple roots α_1 and α_2 of A_2, and the hyperplane H_0 denote the remaining third wall of C_0. Furthermore, let F_0, F_1, F_2 be the corresponding facets of C_0 and $G_0 \in \mathcal{F}_2^{\uparrow}$ denotes the vertex opposite to F_0 (in the 2-simplex C_0). Now fix as x_0 a point in C_0, which is generic with respect to the basis α_1, α_2.

Then set α_1 and α_2 to be the chosen basis of the (character/coroot) lattice Λ and without loss of generality say the segment s_i does not leave C_0 through F_i for $i = 1, 2$ (otherwise take $-\alpha_i$ as base element).

In this way, we choose C_0, x_0 and the basis α_1 and α_2, such that $\overline{\mathcal{F}^{\uparrow}}$ of $\mathscr{A}_2^{\uparrow}$ is constructed without "double crosses" and is thus a set of representatives (see Figure 5.2.2). □

5.3 For Weyl arrangements

Lemma 5.3.1. *Let $\mathscr{A}$ be a toric Weyl arrangement. Then there exist C_0, x_0 and a basis of the coroot lattice, such that the set $\overline{\mathcal{F}^{\uparrow}}$ is a fundamental domain of the action of Λ on $\mathcal{F}^{\uparrow}$.*

Proof. Choose a chamber C_0 of the affine Weyl arrangement $\mathscr{A}^{\uparrow}$. Remember that n is the rank of the Weyl group and $\Lambda \cong \mathbb{Z}^n$. Take n coroots which are orthogonal to facets of C_0 and generate the coroot lattice Λ as basis. Let $\alpha_1, ..., \alpha_n$ denote those coroots, the usual $\vee$ to indicate a coroot is omitted to simplify matters.

This means that the $\mathbb{Z}$-span of $\alpha_1, ..., \alpha_n$ is Λ and there exist corresponding facets $F_1, ..., F_n \in \mathcal{F}_1^{\uparrow}$ of C_0, such that F_i is orthogonal to α_i for all i. Note that this choice of the basis just depends on C_0 and not on x_0, thus it is valid to fix x_0 as an arbitrary generic point relative to $\alpha_1, ..., \alpha_n$.

Without loss of generality the α_i are chosen, such that the segment s_i does not leave C_0 by crossing F_i and thus $\hat{\alpha_i}.s_i$ arrives in $\alpha.C_0$ by crossing the translation $\hat{\alpha_i}.F_i$ of F_i (with $\alpha := \alpha_1 ... \alpha_n$ and $\hat{\alpha_i}$ defined as in (5.1)). Therefore, we avoid that the segments $\hat{\alpha_i}.s_i$ arrive in $\alpha.C_0$ by crossing the same facet (and the chamber "before"). According to Lemma 5.1.5 this would yield more than one representative for some orbits.

Such a basis can always be chosen, since there always exists a total order (on V) such that C_0 is the Weyl chamber (or a translation of it) for the particular simple (respectively positive) system corresponding to this total order (see Section 1.2.1 and Definition 1.3.9).

As above, "double crosses" are avoided by this choice and the set $\overline{\mathcal{F}^{\uparrow}}$ is a set of representatives of the action of Λ on $\mathcal{F}^{\uparrow}$. □

6 Fundamental Group in Rank 2

In Chapter 4 we got a presentation of $\pi_1(\mathcal{M}(\mathscr{A}))$ by Theorem 4.3.6 as final result, namely

$$\pi_1(\mathcal{M}(\mathscr{A})) = \langle \overline{\tau_1}, ..., \overline{\tau_n}; \overline{\gamma_F}, F \in \Omega \cap \overline{\mathcal{F}^{\uparrow}}_1 | \, \overline{\tau_i}\,\overline{\tau_j} = \overline{\tau_j}\,\overline{\tau_i} \text{ for } i, j = 1, ..., n; R_G^{\downarrow}, G \in \overline{\mathcal{F}^{\uparrow}}_2 \rangle.$$

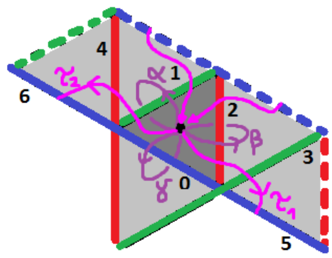
Our conjecture is that the number of generators can be restricted even further to one $\overline{\gamma_F}$ per hypersurface in $\mathscr{A}$. In this chapter we will show that this is true for all toric Weyl arrangements corresponding to Weyl groups of rank two.

6.1 The A_2-case

So consider the toric Weyl arrangement $\mathscr{A}_2^{\uparrow}$ corresponding to A_2, then the fundamental group of $\mathscr{A}_2$ is generated by $\overline{\tau_1}, \overline{\tau_2}$ and six $\overline{\gamma_F}$ by the theorem above, since there exist six faces F in $\Omega \cap \overline{\mathcal{F}^{\uparrow}}$.

Lemma 6.1.1. *The fundamental group $\pi_1(\mathcal{M}(\mathscr{A}_2))$ is generated by the homotopy classes of τ_1, τ_2, and three γ_F, one for each hypersurface of $\mathscr{A}_2$.*

Figure 6.1.2. **Fundamental domain $\overline{\mathcal{F}^{\uparrow}}$ of $\mathscr{A}_2$.**



The faces are indexed by the number i of the corresponding F_i in $\mathscr{A}_2^{\uparrow}$.

Proof. Fix C_0 and the basis of $\Lambda = \langle \Phi^{\vee} \rangle_{\mathbb{Z}} \cong \mathbb{Z}^2$ as in Section 5.2, likewise F_0, F_1, F_2 are the facets of C_0 labelled as above. Furthermore, set x_0 to be a point which is obtained by moving from the barycenter in the direction towards G_0 away from F_0. The point x_0

is generic with respect to α_1, α_2. Thus the set $\overline{\mathcal{F}^\uparrow}$ is a fundamental domain, in fact it is the one seen in Figure 5.2.2.

Now choose three faces G in $\overline{\mathcal{F}^\uparrow}$ and then show that all six γ_F with F in $\Omega \cap \overline{\mathcal{F}^\uparrow}$ are homotopic to a product of their γ_G and the τ_i. Therefore, let F_0, F_1 and F_2 in $\mathcal{F}_1(\mathscr{A}_2^\uparrow)$ denote the facets of C_0 as in Section 5.2 and set

$$\alpha = \gamma_{F_1},\ \beta = \gamma_{F_2} \text{ and } \gamma = \gamma_{F_0}.$$

Thus it follows by definition of γ_F (and the fact of the F_i lying in $\overline{\mathcal{F}^\uparrow}$) that

$$\alpha = p(\beta_{F_1}),\ \beta = p(\beta_{F_2}) \text{ and } \gamma = p(\beta_{F_0}).$$

The six faces in $\Omega \cap \overline{\mathcal{F}^\uparrow}$ are denoted in the following way (compare Figure 6.1.2):

- The first two faces in $\Omega \cap \overline{\mathcal{F}^\uparrow}$ are already labelled by F_1 and F_2.
- Then let F_3 denote the face whose support is a translation of the hyperplane $\operatorname{supp}(F_1)$ and F_4 denote the face whose support is a translation of $\operatorname{supp}(F_2)$.
- F_5 and F_6 are the faces with the same support as F_0.

Furthermore, set

$$\alpha' = \gamma_{F_3},\ \beta' = \gamma_{F_4},\ \gamma' = \gamma_{F_5} \text{ and } \gamma'' = \gamma_{F_6}.$$

Therefore, the path α' (respectively β' and γ',γ'') makes a loop around the same hypersurfaces as α (respectively β and γ) in $\mathcal{M}(\mathscr{A}_2)$. Moreover, let the liftings of these paths to $\mathcal{M}(\mathscr{A}_2^\uparrow)$ be denoted by the uparrow $^\uparrow$ (all starting in C_0).

Let G denote the face in $\overline{\mathcal{F}^\uparrow}_1 \backslash \Omega$ with $\operatorname{supp}(G) = \operatorname{supp}(F_3)$. Then we deduce in $\mathcal{M}(\mathscr{A}_2)$:

$$(\alpha')^\uparrow \overset{4.1.8}{\simeq} \beta^\uparrow \beta_G (\beta^\uparrow)^{-1} \overset{4.1.8}{\simeq} \beta^\uparrow \omega_2^{-1} u_2^{-1}.\alpha^\uparrow u_2^{-1}.\omega_2(\beta^\uparrow)^{-1}.$$

Hence,

$$\alpha' \simeq \beta \tau_2^{-1} \alpha \tau_2 \beta^{-1}.$$

Similiarly it follows

$$\beta' \simeq \alpha \tau_1^{-1} \beta \tau_1 \alpha^{-1}.$$

It is clear that

$$(\gamma')^\uparrow \overset{4.1.8}{\simeq} (\gamma'')^\uparrow \overset{4.1.8}{\simeq} \gamma^\uparrow.$$

Thus for the last reamaining hypersurfaces we get

$$\gamma' \simeq \gamma'' \simeq \gamma.$$

Therefore, the fundamental group of $\mathcal{M}(\mathscr{A}_2)$ can be presented as

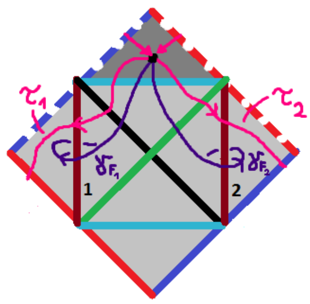
$$\pi_1(\mathcal{M}(\mathscr{A}_2)) = \langle \overline{\tau_1}, \overline{\tau_2}; \overline{\alpha}, \overline{\beta}, \overline{\gamma} | \tau_1\tau_2 \simeq \tau_2\tau_1; R_G^{\downarrow}, G \in \overline{\mathcal{F}^{\uparrow}}_2 \rangle.$$

□

6.2 The BC_2-case

Lemma 6.2.1. *Let $\mathscr{A}$ be the toric Weyl arrangement corresponding to BC_2. Then the fundamental group $\pi_1(\mathcal{M}(\mathscr{A}))$ is generated by the homotopy classes of τ_1, τ_2 and six γ_F, one for each hypersurface in $\mathscr{A}$.*

Figure 6.2.2. **Fundamental domain $\overline{\mathcal{F}^{\uparrow}}$ for BC_2.**



The faces are indexed by the number i of the corresponding F_i in $\mathscr{A}^{\uparrow}$.

Proof. Let C_0 be an arbitrary chamber in $\mathscr{A}^{\uparrow}$. Then let α_1, α_2 be the coroots orthogonal to the facets of C_0 which are the two legs of C_0 regarded as a right-angled triangle. As above, the coroots α_1, α_2 (in C_2 or B_2 depending if you started with B_2 or C_2) should be chosen, such that for any point x_0 in C_0 the segments s_1, s_2 do not leave C_0 by meeting their orthogonal facet. By construction, the coroots α_1, α_2 generate Λ. By this choice, any point in C_0 is generic. So fix some point x_0 in C_0.

For this construction of $\overline{\mathcal{F}^{\uparrow}}$ the set $\Omega \cap \overline{\mathcal{F}^{\uparrow}}_1$ contains seven faces, but the toric Weyl arrangement $\mathscr{A}$ corresponding to BC_2 consists of six hypersurfaces. Let F_1 and F_2 denote the faces in $\Omega \cap \overline{\mathcal{F}^{\uparrow}}_1$ which are orthogonal to the long edge of C_0. Then $p(F_1)$ and $p(F_2)$

lie on the same hypersurface of $\mathscr{A}$. Without loss of generality let F_1 be the face met by s_1. Thus by construction and Lemma 4.1.8 holds

$$\beta_{F_2} \simeq \omega_1^{-1} u_1^{-1}.\beta_{F_1}\omega_1.$$

Hence

$$\gamma_{F_2} = p(\beta_{F_2}) \simeq \tau_1^{-1}\gamma_{F_1}\tau_1.$$

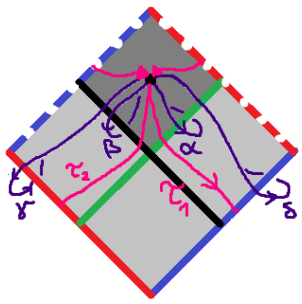
Therefore, the claim follows by Theorem 4.3.6. □

6.3 The D_2-case

Lemma 6.3.1. *Let $\mathscr{A}$ be the toric Weyl arrangement corresponding to the reducible group $D_2 = A_1 \oplus A_1$. Then the fundamental group $\pi_1(\mathcal{M}(\mathscr{A}))$ is generated by the homotopy classes of τ_1, τ_2, and four γ_F, one for each hypersurface in $\mathscr{A}$.*

Figure 6.3.2.

Fundamental domain $\overline{\mathcal{F}^{\uparrow}}$ for D_2.



Proof. Let C_0 be an arbitrary chamber of $\mathscr{A}^{\uparrow}$, $\alpha_1, \alpha_2 \in D_2^{\vee} = D_2$ a basis of the coroot lattice Λ, and x_0 an arbitrary point in C_0 (any point in C_0 is generic).

Then the set $\Omega \cap \overline{\mathcal{F}^{\uparrow}}_1$ contains four faces, whereat each one corresponds to a different hypersurface in $\mathscr{A}$. Thus the claim follows immediately by Theorem 4.3.6. □

Furthermore, we get as relations $R_G^{\downarrow}$:

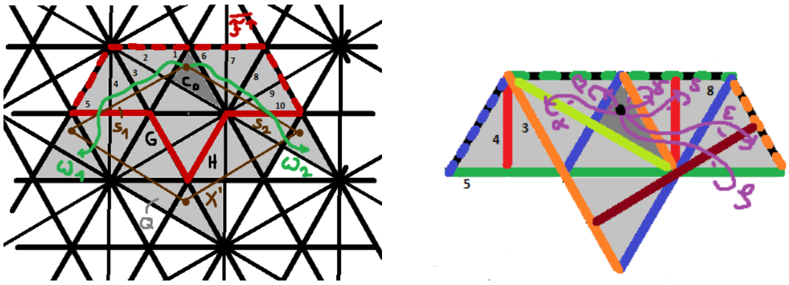
$$\alpha\beta \simeq \beta\alpha, \alpha\gamma \simeq \gamma\alpha, \beta\delta \simeq \delta\beta \text{ and } \gamma\delta \simeq \delta\gamma,$$

where $\alpha, \beta, \gamma, \delta$ are denoted as in Figure 6.3.2.

6.4 The G_2-case

Lemma 6.4.1. *Let $\mathscr{A}$ be the toric Weyl arrangement corresponding to G_2. Then the fundamental group $\pi_1(\mathcal{M}(\mathscr{A}))$ is generated by the homotopy classes of τ_1, τ_2 and six γ_F, one for each hypersurface in $\mathscr{A}$.*

Figure 6.4.2. **Fundamental domain $\overline{\mathcal{F}^{\uparrow}}$ for G_2.**



The faces are indexed by the number i of the corresponding F_i in $\mathscr{A}^{\uparrow}$.

Proof. Choose an arbitrary chamber C_0 of $\mathscr{A}^{\uparrow}$. As above C_0 can be regarded as a right-angled triangle. Then choose element the root α_1 in $G_2^{\vee}$ which is orthogonal to the hypotenuse of C_0 and directed in a way such that s_1 does not leave C_0 by meeting the facet which is the hypotenuse as the first basis element. The second basis element α_2 should be the root in $G_2^{\vee}$ which is orthogonal to the short leg of C_0 and directed in a way such that s_2 does not leave C_0 by meeting the facet which is the short leg.

As generic point x_0 fix a point close to the vertex between the short leg and the hypotenuse. For example, choose the point which lies on three quarters of the segment from the barycenter to this vertex. By this choice, the set $\overline{\mathcal{F}^{\uparrow}}$ is a fundamental domain and $\Omega \cap \overline{\mathcal{F}^{\uparrow}}_1$ contains ten faces.

Label this faces in the following way:

- Let F_1, F_2, F_3, F_4, F_5 be the faces met by s_1 (ordered in the way they are met).
- And let $F_6, F_7, F_8, F_9, F_{10}$ be the faces met by s_2 (ordered as well in the way they are met).

The toric Weyl arrangement $\mathscr{A}$ corresponding to G_2 contains six hypersurfaces, thus we have to reduce the number of generators γ_{F_i} (corresponding to the faces $F_1, ..., F_{10}$) by four.

So set

$$\alpha = \gamma_{F_2}, \beta = \gamma_{F_1}, \gamma = \gamma_{F_6}, \delta = \gamma_{F_7}, \varepsilon = \gamma_{F_9} \text{ and } \zeta = \gamma_{F_{10}}.$$

Then it follows for β_{F_3} in $\mathcal{M}(\mathscr{A}^\uparrow)$, that

$$\beta_{F_3} \overset{4.1.8}{\simeq} \beta_{F_2}\beta_{F_1}\beta_G(\beta_{F_2}\beta_{F_1})^{-1} \overset{4.1.8}{\simeq} \beta_{F_2}\beta_{F_1}\omega_2^{-1}u_2^{-1}.\beta_{F_6}\omega_2(\beta_{F_2}\beta_{F_1})^{-1},$$

where G denotes a face in $\overline{\mathcal{F}^\uparrow}$ which is not equal to F_3, but has the same support.

Hence, we deduce

$$\gamma_{F_3} \simeq \alpha\beta\tau_2^{-1}\gamma\tau_2(\alpha\beta)^{-1}.$$

For the face F_4 it holds that

$$\beta_{F_4} \overset{4.1.8}{\simeq} \omega_1(u_1.\beta_{F_6})^{-1}u_1.\beta_{F_7}u_1.\beta_{F_6}\omega_1^{-1}.$$

Therefore, it follows

$$\gamma_{F_4} \simeq \tau_1\gamma^{-1}\delta\gamma\tau_1^{-1}.$$

Moreover, for the face F_8 we get

$$\beta_{F_8} \overset{4.1.8}{\simeq} \beta_{F_7}\beta_{F_6}\beta_H(\beta_{F_7}\beta_{F_6})^{-1} \overset{4.1.8}{\simeq} \beta_{F_7}\beta_{F_6}\omega_1^{-1}u_1^{-1}.\beta_{F_1}\omega_1(\beta_{F_7}\beta_{F_6})^{-1},$$

where H denotes the face in $\overline{\mathcal{F}^\uparrow}$ which is not equal to F_8, but has the same support.

Thus for γ_{F_8} in $\mathcal{M}(\mathscr{A})$ follows

$$\gamma_{F_8} \simeq \delta\gamma\tau_1^{-1}\beta\tau_1(\delta\gamma)^{-1}.$$

Finally, for the last remaining face F_5 we have

$$\beta_{F_5} \overset{4.1.8}{\simeq} (\beta_{F_9}\beta_{F_8}\beta_{F_7}\beta_{F_6})^{-1}\beta_{F_{10}}(\beta_{F_9}\beta_{F_8}\beta_{F_7}\beta_{F_6}).$$

Hence, in $\mathcal{M}(\mathscr{A})$ it holds

$$\gamma_{F_5} \simeq (\varepsilon\,\gamma_{F_8}\delta\,\gamma)^{-1}\zeta(\varepsilon\,\gamma_{F_8}\delta\,\gamma).$$

Therefore, the fundamental group $\pi_1(\mathcal{M}(\mathscr{A}))$ is generated by the homotopy classes of $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ and τ_1, τ_2. □

7 Going Forward

1. Higher rank.

For the arrangements corresponding to Weyl groups of rank two it is true, that the generators of the fundamental group of the complement of a toric Weyl arrangement $\mathscr{A}$ can be reduced to the generators of the fundamental group of the torus and one generator per hypersurface in $\mathscr{A}$.

However, for Weyl groups of higher rank the question remains open, even in the case of the arrangement $\mathscr{A}_n$. The main problem is the choice of representatives for the hypersurfaces, such that they generate the fundamental group.

2. Fundamental domain.

Moreover, for the first reduction of the generators the set $\overline{\mathcal{F}^{\uparrow}}$ at the end of Chapter 4 has to be a set of representatives as discussed in Chapter 5. For toric Weyl arrangements the base chamber C_0, the base point x_0 and the basis elements $u_1, ..., u_n$ of the character lattice can be chosen in such a way that $\overline{\mathcal{F}^{\uparrow}}$ is a fundamental domain.

This yields the challenge to choose C_0, x_0 and the u_i in a more general way such that $\overline{\mathcal{F}^{\uparrow}}$ is a fundamental domain for an arbitrary toric arrangement. A possible strategy could be to use vectors orthogonal to the hyperplanes in $\mathscr{A}^{\uparrow}$ similarly to the idea with orhtogonal coroots for Weyl arrangement from Chapter 5. Alternatively, one can try to adopt the construction of [2].

Bibliography

[1] Giacomo d'Antonio and Emanuele Delucchi: *A Salvetti Complex for Toric Arrangements and its fundamental group*; ArXiv e-print 1101.4111, February 2011.

[2] Giacomo d'Antonio and Emanuele Delucchi: *Minimality of toric arrangements*; ArXiv e-print 1112.5041, December 2011.

[3] Nicolas Bourbaki: *Lie Groups and Lie Algebras: Chapter 4-6*; Springer, Berlin, 2002.

[4] Alexandre V. Borovik and Anna Borovik: *Mirrors and Reflections*; Springer, New York, 2009.

[5] Corrado De Concini and Claudio Procesi: *Topics in Hyperplane Arrangements, Polytopes and Box-Splines*; Springer Verlag, Berlin, 2010.

[6] Richard Ehrenborg, Margaret Readdy and Michael Slone: *Affine and toric hyperplane arrrangements*; ArXiv e-print 0810.0295v1, October 2008.

[7] Allen Hatcher: *Algebraic Topology*; Cambridge University Press, Cambridge, 2001.

[8] James E. Humphreys: *Reflection Groups and Coxeter Groups*; Cambridge University Press, Cambridge, 1990.

[9] James E. Humphreys: *Introduction to Lie Algebras and Representation Theory*; Springer, New York, 1972.

[10] Dmitry Kozlov: *Combinatorial Algebraic Topology*; Springer, Berlin, 2008.

[11] Gus I. Lehrer: *A toral configuration space and regular semisimple conjugacy classes*; Mathematical Proceedings of the Cambridge Philosophical Society 118, pp. 105Ũ13, 1995.

[12] Luca Moci and Simona Settepanella: *The homotopy type of toric arrangements*; Journal of Pure and Applied Algebra 215, pp. 1980-1989, 2011.

[13] Luca Moci: *Wonderful models of toric arrangements*; ArXiv e-print 0912.5461v2, February 2011.

[14] Peter Orlik and Hiroaki Terao: *Arrangements of Hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*; Springer, Berlin, 1992.

[15] Mario Salvetti: *Topology of the complement of real hyperplanes in* $\mathbb{C}^N$; Inventiones mathematicae 88(3), pp. 603-618, 1987.