Universität Bremen

Bachelor Thesis

Brylawski's Conjecture and the Arithmetic Tutte Polynomial

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July 27, 2012

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Declaration of Authorship

I hereby certify that this thesis has been composed by me and is based on my own work, unless stated otherwise. No other person's work has been used without due acknowledgement in this thesis. All references have been quoted and all sources of information have been specifically acknowledged.

Bremen, July 27, 2012 Julia Heller

1 Introduction

Brylawski conjectured 1972 in [2] that the Tutte polynomial of a connected matroid is irreducible. This assumption was proven by Merino, de Mier and Noy 2001 in [5]. The main tool of this proof is a set of linear equations based on the coefficients of the Tutte polynomial. The subject of the present thesis is a complete account of the proof of the irreducibility of the Tutte polynomial of a connected matroid. In particular, the precise conditions required for the validity of the set of linear equations were brought into focus. Thereby some proofs turned out to be incomplete or even wrong. These proofs were corrected and rewritten with a different approach.
This analysis is the foundation of the study weather Brylawski's conjecture could also be generalized to the arithmetic Tutte polynomial. We leave this as a future object of investigation.

2 Definitions and Basics

All following definitions can be found in [2], [5] and [6].

Definition. A *matroid* $M = (E, \mathcal{I})$ is an ordered pair consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E with the following properties:

($\mathcal{I}$1) $\emptyset \in \mathcal{I}$.

($\mathcal{I}$2) If $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$.

($\mathcal{I}$3) If $I, J \in \mathcal{I}$ and $|I| < |J|$, then an element $e \in J \setminus I$ exists, such that $I \cup e \in \mathcal{I}$.

The set E is called the *ground set* of M and the elements of $\mathcal{I}$ or $\mathcal{I}(M)$ are the *independent sets*. A set not contained in $\mathcal{I}$ is called *dependent*.

In the following definitions let $M = (E, \mathcal{I})$ be a matroid.

Definition. A maximal independent set of M is a *basis* of M, $\mathcal{B}(M)$ denotes the set of bases of M.

Definition. The *dual matroid* M^* of M is a matroid with ground set E and bases $\mathcal{B}(M^*) = \{E \setminus B : B \in \mathcal{B}(M)\}$. For $B \in \mathcal{B}(M)$ the *dual basis* of B is denoted by B^*.

Definition. A minimal dependent set of M is called a *circuit* of M.

Definition. For $A \subseteq E$ the *rank* of A in M is defined as the cardinality of a greatest independent set contained in A, $r_M(A) := \max_{X \subseteq A, X \in \mathcal{I}(M)} |X|$.
The *rank of a matroid* M is $r_M(M) := r_M(E)$.
The *corank* of A in M is $cr_M(A) := r_M(M) - r_M(A)$ and the *nullity* of A in M is $n_M(A) := |A| - r_M(A)$.

Remark. The rank of a matroid is the cardinality of a basis.

Corollary 1. *A subset $A \subseteq E$ is independent in M if and only if $n_M(A) = 0$.*

Definition. The *closure* of $A \subseteq E$ in M is $cl_M(A) := \{x \in E : r_M(A \cup x) = r_M(A)\}$.

Definition. $F \subseteq E$ is a *flat* or *closed set* of M if $cl_M(F) = F$.

Definition. A flat $H \subseteq E$ with $r_M(H) = r_M(M) - 1$ is called a *hyperplane* of M and the set $E \setminus H$ is called a *cocircuit* of M.

Remark. If $C \subseteq E$ is a cocircuit in M, then C is a circuit in M^*.

Definition. An element $e \in E$, for which $\{e\}$ is a circuit of M is called a *loop*. An element of E, which is in every basis of M is an *isthmus*. An element of E that is neither a loop nor an isthmus is called *nonfactor* of M.

Remark. Loops of M are not in any basis of M.

Definition. For $e \in E$ the *deletion* $M \setminus e$ is a matroid $M \setminus e = (E \setminus e, \mathcal{I}(M \setminus e))$, where the independent sets are

$$\mathcal{I}(M \setminus e) = \{I \in \mathcal{I}(M) : e \notin I\}.$$

Definition. If $e \in E$ is not a loop, the *contraction* M/e is a matroid $M/e = \{E \setminus e, \mathcal{I}(M/e)\}$ with independent sets

$$\mathcal{I}(M/e) = \{I : I \subseteq B \setminus e, B \in \mathcal{B}(M), e \in B\}.$$

When $e \in E$ is a loop, the contraction M/e is defined as the deletion $M \setminus e$.

Definition. For two matroids $M_1 = (E_1, \mathcal{I}(M_1))$ and $M_2 = (E_2, \mathcal{I}(M_2))$ with $E_1 \cap E_2 = \emptyset$ the *direct sum* of M_1 and M_2 is a matroid $M_1 \oplus M_2 = (E_1 \cup E_2, \mathcal{I}(M_1 \oplus M_2))$ with independent sets

$$\mathcal{I}(M_1 \oplus M_2) = \{I_1 \cup I_2 : I_1 \in \mathcal{I}(M_1), I_2 \in \mathcal{I}(M_2)\}.$$

Definition. For a matroid $M = (E, \mathcal{I})$ the *Tutte polynomial* is defined as

$$T(M; x, y) = \sum_{A \subseteq E} (x-1)^{cr_M(A)} (y-1)^{n_M(A)}.$$

Proposition 1 (stated in [5, Theorem 1.4.]). *Let the matroid M be the direct sum of M_1 and M_2. Then*

$$T(M; x, y) = T(M_1; x, y) T(M_2; x, y).$$

Definition. A matroid M is *connected* if it cannot be expressed as a direct sum of two nonempty matroids.

Proposition 2 (stated in [5]). *M is connected if and only if for every two distinct elements of E there is a circuit containing both.*

Remark. Every matroid with $|E| = 1$ is connected since it cannot be expressed as a direct sum of two nonempty matroids.

Remark. For $|E| > 1$, the smallest connected matroid is $M = (E, \mathcal{I})$ with $E = \{1, 2\}$ and $\mathcal{I} = \{\emptyset, \{1\}, \{2\}\}$. Its Tutte polynomial is

$$\begin{aligned}
T(M; x, y) &= \sum_{A \subseteq E} (x-1)^{cr(A)} (y-1)^{n(A)} \\
&= (x-1)^{cr(\emptyset)} (y-1)^{n(\emptyset)} + (x-1)^{cr(\{1\})} (y-1)^{n(\{1\})} \\
&\quad + (x-1)^{cr(\{2\})} (y-1)^{n(\{2\})} + (x-1)^{cr(E)} (y-1)^{n(E)} \\
&= (x-1)^1 (y-1)^0 + (x-1)^0 (y-1)^0 + (x-1)^0 (y-1)^0 + (x-1)^0 (y-1)^1 \\
&= x - 1 + 1 + 1 + y - 1 \\
&= x + y.
\end{aligned}$$

M is the only connected matroid on a ground set with two elements.

Definition. A *Boolean algebra* is a matroid B_n, which is the direct sum of n isthmuses. A *pre-Boolean algebra* is a matroid B_{nm}, which is the direct sum of n isthmuses and m loops.

Remark. Since all elements of B_{nm} are either loops or isthmuses, B_{nm} is the general matroid with zero nonfactors.

Proposition 3 (stated in [2]). *The Tutte polynomials of B_n and B_{nm} are $T(B_n; x, y) = x^n$ and $T(B_{nm}; x, y) = x^n y^m$ respectively.*

Theorem 1 (**Deletion-Contraction Formula**, stated in [3, Theorem 6.2.2.]). *If $e \in E$ is a nonfactor of the matroid $M = (E, \mathcal{I})$, then*

$$T(M; x, y) = T(M \setminus e; x, y) + T(M/e; x, y).$$

Proposition 4. *For a matroid M, $T(M; 1, 1)$ enumerates the bases of M.*

Proof. $T(M; 1, 1) = \sum_{A \subseteq E}(1-1)^{cr_M(A)}(1-1)^{n_M(A)} = \sum_{B \subseteq E} 1$, where $B \subseteq E$ are those subsets of E, for which $cr_M(B) = n_M(B) = 0$. These subsets B are exactly the bases of M. □

Basis Activities (adapted from [3, Section 6.6.A] and [2, Section 6]).
Let M be a matroid. We want to describe its Tutte polynomial in an alternative form $\sum_{i,j} b_{ij}x^i y^j$. Therefore we consider the *internal* and *external activity* of the bases of M. Since $T(M; 1, 1)$ enumerates the bases of M by Proposition 4, we may partition $\mathcal{B}(M)$ into blocks $\mathcal{B}_{ij}$. Then we set $b_{ij} := |\mathcal{B}_{ij}|$ as the coefficient of $x^i y^j$ in the Tutte polynomial $T(M; x, y)$.
To obtain such a partition, we first linearly order the ground set E of M by relabeling its elements by $1, \dots, n$.
Let $B \in \mathcal{B}(M)$. An element $p \in B$ is *internally active*, if p is the least element in the unique cocircuit contained in $(E \setminus B) \cup p$. The number of elements of B, which are internally active is called the *internal activity* $\iota(B)$ of B.
An element $q \in E \setminus B$ is *externally active*, if q is the least element in the unique circuit contained in $B \cup q$. The number of elements of $E \setminus B$, which are externally active is called the *external activity* $\varepsilon(B)$ of B.
We define b_{ij} as the number of bases with internal activity i and external activity j to get the desired partition of $\mathcal{B}(M)$. This is

$$\mathcal{B}_{ij} = \{B \in \mathcal{B}(M) : \iota(B) = i, \varepsilon(B) = j\}.$$

The coefficients b_{ij} do not depend on the linear ordering chosen for E.

Remark. Since b_{ij} counts bases of M it is $b_{ij} \geq 0$ for all i, j.

Proposition 5. *Let M be a matroid and M^* its dual. Then $\iota(B) = \varepsilon(B^*)$ and $\varepsilon(B) = \iota(B^*)$ for all $B \in \mathcal{B}(M)$.*

Proof. Let B a Basis of M with an internally active element $p \in B$. Then by definition p is the least element in the unique cocircuit contained in $(E \setminus B) \cup p = B^* \cup p$. Since a cocircuit of M is a circuit in M^*, $p \in E \setminus B^*$ is the least element in the unique circuit contained in $B^* \cup p$ and hence externally active in M^*. This implies the equality $\iota(B) = \varepsilon(B^*)$. Since the dual of M^* is M it follows that $\varepsilon(B) = \iota(B^*)$. □

Corollary 2 (stated in [5, Theorem 1.6.]). *If $T(M; x, y)$ is the Tutte polynomial of a matroid M, then $T(M^*; y, x)$ is the Tutte polynomial of its dual M^*.*

Corollary 3. *A matroid is connected if and only if its dual is connected.*

Definition. Let $M = (E, \mathcal{I})$ be a matroid. Its Tutte polynomial via basis activities is

$$T(M; x, y) = \sum_{i,j} b_{ij} x^i y^j.$$

3 Brylawski's Conjecture

Theorem 2 (stated in [4, Theorem 1]). *Let M be a connected matroid. Then its Tutte polynomial $T(M; x, y)$ is irreducible in $\mathbb{Z}[x, y]$.*

4 Preliminary Theorems

Proposition 6 (adapted from [2, Proposition 4.1.]). *Let $M = (E, \mathcal{I})$ be a matroid and $e \in E$, then the following statements apply.*

(1) For $A \subseteq E \setminus e$ is $r_M(A) = r_{M\setminus e}(A)$.

(2) If $e \in E$ is not a loop of M and $e \in A \subseteq E$, then $r_M(A) = r_{M/e}(A \setminus e) + 1$.

(3) $e \in E$ is an isthmus of M if and only if $r_M(M) = r_M(E \setminus e) + 1 = r_{M\setminus e}(M \setminus e) + 1$. If $e \in E$ is not an isthmus of M, then $r_M(M) = r_{M\setminus e}(M \setminus e)$.

(4) $e \in E$ is a loop of M if and only if $r_M(M) = r_{M/e}(M/e)$. If $e \in E$ is not a loop of M, then $r_M(M) = r_{M/e}(M/e) + 1$.

Proof. (1) For $r_M(A) = \max_{X \in \mathcal{I}, X \subseteq A} |X|$ it suffices considering $X \in \{I \in \mathcal{I}(M) : e \notin I\} = \mathcal{I}(M \setminus e)$, since $e \notin A$ implies $e \notin X$. Hence we get

$$\begin{aligned} r_M(A) &= \max_{X \subseteq A, X \in \mathcal{I}(M)} |X| \\ &= \max_{X \subseteq A, X \in \mathcal{I}(M\setminus e)} |X| \\ &= r_{M\setminus e}(A). \end{aligned}$$

(2) It is $r_M(A) = \max_{X \subseteq A, X \in \mathcal{I}(M)} |X| = |B|$ for a $B \in \mathcal{I}(M)$. Since $e \in E$ is not a loop of M and $e \in A$, e is not a loop of A and one can choose B such that $e \in B$. Since the independent sets of the contraction are $\mathcal{I}(M/e) = \{I \subseteq C \setminus e : C \in \mathcal{B}(M), e \in C\}$, it follows that $B \setminus e \in \mathcal{I}(M/e)$. Moreover, $|B \setminus e| = \max_{X \subseteq A \setminus e, X \in \mathcal{I}(M/e)} |X| = r_{M/e}(A \setminus e)$. So we get

$$\begin{aligned} r_M(A) &= |B| \\ &= |B \setminus e| + |e| \\ &= r_{M/e}(A \setminus e) + 1. \end{aligned}$$

(3) If $e \in E$ is an isthmus of M, then for every basis B of M, $B \setminus e$ is the greatest independent set of M contained in $E \setminus e$. The converse also holds if e is an isthmus. If e is not an isthmus, it is not in every basis of M and thus one can find a basis of M contained in $E \setminus e$.
Since the cardinality of the greatest independent set of M contained in $E \setminus e$ is $|B \setminus e| = |B| - 1$ if e is an isthmus, it is $r_M(E \setminus e) = r_M(M) - 1$. The assertion follows with part (1).
If e is not an isthmus of M, then the above mentioned equicardinality of the bases of M and $M \setminus e$ implies $r_M(M) = r_{M \setminus e}(M \setminus e)$.

(4) An element $e \in E$ is a loop of M if and only if it is not contained in any basis of M. If $e \in E$ is a loop of M, the contraction M/e is defined as the deletion $M \setminus e$. In this case it is $\mathcal{B}(M/e) = \mathcal{B}(M \setminus e) = \{B \in \mathcal{B}(M) : e \notin B\} = \mathcal{B}(M)$ and it follows $r_M(M) = r_{M/e}(M/e)$ if and only if e is a loop of M.
If e is not a loop of M, the bases of M/e are $\mathcal{B}(M/e) = \{B \setminus e : B \in \mathcal{B}(M), e \in B\}$ and thus $r_{M/e}(M/e) = r_M(M) - 1$.

□

Lemma 1 (concluded from [2, Proposition 4.1.]). *Let $M = (E, \mathcal{I})$ be a matroid. Then $F \subseteq E \setminus e$ is a flat in M/e if and only if $F \cup e$ is a flat in M.*

Proof. We first assume that $e \in E$ is not a loop of M. With part (2) of Proposition 6 it follows that

$$\begin{aligned} F = cl_{M/e}(F) &= \{x \in E \setminus e : r_{M/e}(F \cup x) = r_{M/e}(F)\} \\ &= \{x \in E \setminus e : r_M(F \cup e \cup x) = r_M(F \cup e)\} \\ &= cl_M(F \cup e) \setminus e, \end{aligned}$$

which is equal to

$$
\begin{aligned}
F \cup e &= \{x \in E : r_M(F \cup e \cup x) = r_M(F \cup e)\} \\
&= cl_M(F \cup e).
\end{aligned}
$$

We now assume that $e \in E$ is a loop of M. Since in this case M/e is defined as $M \setminus e$ it follows with part (1) of Proposition 6 that

$$
\begin{aligned}
F = cl_{M/e}(F) = cl_{M\setminus e}(F) &= \{x \in E \setminus e : r_{M\setminus e}(F \cup x) = r_{M\setminus e}(F)\} \\
&= \{x \in E \setminus e : r_M(F \cup x) = r_M(F)\}.
\end{aligned}
$$

Since e is a loop of M and hence it is not in any independent set, this equals

$$
\begin{aligned}
F \cup e &= \{x \in E : r_M(F \cup x) = r_M(F)\} \\
&= \{x \in E : r_M(F \cup e \cup x) = r_M(F \cup e)\} \\
&= cl_M(F \cup e)
\end{aligned}
$$

□

Corollary 4. *It is* $n_{M\setminus e}(M \setminus e) = n_M(M) - 1$ *and* $n_{M/e}(M/e) = n_M(M)$.

Corollary 5. *Let* $M = (E, \mathcal{I})$ *be a matroid and* $e \in E$ *a nonfactor of* M*. Then all* p*-element subsets of* E *are independent in* M *if and only if all* p*-element subsets of* $E \setminus e$ *are independent in* $M \setminus e$ *and all* $(p-1)$*-element subsets of* $E \setminus e$ *are independent in* M/e*.*

Proof. By Proposition 6.(1), for $A \subseteq E \setminus e$ with $|A| = p$ it is $r_M(A) = r_{M\setminus e}(A)$, thus A is independent in M if and only if it is independent in $M \setminus e$. If $e \in A \subseteq E$ with $|A| = p$ then A is independent in M if and only if $A \setminus e$ is independent in M/e, since by Proposition 6.(2) it is $p = r_M(A) = r_{M/e}(A \setminus e) + 1$ if and only if $r_{M/e}(A \setminus e) = p - 1$. □

Lemma 2 (**Basic Properties**, adapted in a corrected form from [4])**.** *Let* M *be a matroid and* $T(M; x, y) = \sum_{i,j} b_{ij} x^i y^j$ *its Tutte polynomial. Then the following basic properties hold:*

(1) $b_{00} = 0$ *if* $|E| \geq 1$

(2) If $|E| \geq 2$ *then* $b_{10} > 0$ *if and only if* M *is connected.*

(3) If M *is connected and* $|E| \geq 2$ *then neither* x *nor* y *are factors of* $T(M; x, y)$*.*

(4) If M is connected then in $T(M;x,y)$ it is

(i)

$$b_{ij} = \begin{cases} 0 & i \geq r(M), j \geq 1 \text{ or } i > r(M), j = 0 \\ 1 & i = r(M), j = 0 \end{cases}$$

(ii)

$$b_{ij} = \begin{cases} 0 & i \geq 1, j \geq n(M) \text{ or } i = 0, j > n(M) \\ 1 & i = 0, j = n(M). \end{cases}$$

Proof. If e is a nonfactor of an arbitrary matroid M then by Theorem 1 it is $T(M;x,y) = T(M \setminus e;x,y) + T(M/e;x,y)$. Let the coefficients of $T(M \setminus e;x,y)$ and $T(M/e;x,y)$ be b'_{ij} and b''_{ij} respectivley.

(1) For an inductive proof we consider a matroid M with $E = \{e\}$. Then e is either (i) a loop or (ii) an isthmus, hence the Tutte polynomial is (i) $T(M;x,y) = (x-1)^{cr(\emptyset)}(y-1)^{n(\emptyset)} + (x-1)^{cr(e)}(y-1)^{n(e)} = y$ or analogously (ii) $T(M;x,y) = x$. In both cases the constant term equals zero.
Assume that $|E| = n$ and that there is a nonfactor e of M. Then $b_{00} = b'_{00} + b''_{00}$ and since the deletion and the contraction are defined on the ground set $E \setminus e$, the induction hypothesis holds for b'_{00} and b''_{00} and thus $b_{00} = 0$.
If M does not contain a nonfactor, it is a pre-Boolean algebra B_{nm} with Tutte polynomial $x^n y^m$ by Proposition 3, which means that $b_{00} = 0$.

(2) Let M be connected and $|E| = 2$. The only matroid satisfying the conditions has the Tutte polynomial $T(M;x,y) = x + y$ by the Remark after Proposition 2. Since $b_{10} = 1 > 0$, the base clause is proven.
Now assume that M is a connected matroid with $|E| > 2$. Since M is connected there is a nonfactor in E and one may apply the Deletion-Contraction Formula such that $T(M;x,y) = T(M \setminus e;x,y) + T(M/e;x,y)$. Since the ground sets of $M \setminus e$ and M/e have size $|E| - 1$, the induction hypothesis holds for the latter and $b'_{10}, b''_{10} > 0$. From this it follows that $b_{10} = b'_{10} + b''_{10} > 0$.
We prove the other direction by contraposition. Therefore let M be a matroid with $|E| \geq 2$, which is not connected. We show that this implies $b_{10} = 0$. Since M is not connected it is the direct sum of two matroids M_1 and M_2 with ground sets ≥ 1. Hence the Tutte polynomial of M is $T(M;x,y) = T(M_1;x,y)T(M_2;x,y)$ by Proposition 1. Let $b^{(1)}_{ij}$ and $b^{(2)}_{ij}$ be the Tutte coefficients of M_1 and M_2 respectively,

then

$$\begin{aligned} b_{10} &= b_{10}^{(1)} \cdot b_{00}^{(2)} + b_{00}^{(1)} \cdot b_{10}^{(2)} \\ &= b_{10}^{(1)} \cdot 0 + 0 \cdot b_{10}^{(2)} \\ &= 0, \end{aligned}$$

since by (1) it is $b_{00}^{(1)} = b_{00}^{(2)} = 0$.

(3) Let M be a connected matroid with $|E| \geq 2$. That implies on the one hand by (2) that $b_{10} > 0$ and on the other hand by Corollary 3 that the dual M^* is connected. By Corollary 2 it is $T(M; x, y) = T(M^*; y, x)$ and thus $b_{01} = b_{10} > 0$. From this it follows that neither x nor y are factors of $T(M; x, y)$, since the terms x and y with coefficients unequal zero appear in the Tutte polynomial.

(4) We make an inductive proof on the size of the ground set of the connected matroid M. For the base clause we consider the two matroids with ground set $\{e\}$, namely $M_1 = (\{e\}, \{e\})$ and $M_2 = (\{e\}, \emptyset)$. Since $M_1 = B_1$, the Boolean algebra with one isthmus the Tutte polynomial is $T(M_1; x, y) = x$. Analogously it is $M_2 = B_{0,1}$ and hence $T(M_2; x, y) = y$. With $r(M_1) = n(M_2) = 1$ and $r(M_2) = n(M_1) = 0$ the Tutte coefficients fulfil the required conditions thus we may now assume that M is a connected matroid with $|E| \geq 2$. As M is a connected matroid with more than one element, there exists a nonfactor of M. Thus we may apply the Deletion-Contraction Formula to $T(M; x, y)$. We get $T(M; x, y) = T(M\backslash e; x, y) + T(M/e; x, y)$ and hence $b_{ij} = b'_{ij} + b''_{ij}$.

(i) First we consider the case $i \geq r_M(M)$ and $j \geq 1$. Then with Proposition 6 by induction hypothesis it is $b'_{ij} = 0$ for $i \geq r_{M\backslash e}(M \setminus e) = r_M(M)$ and $j \geq 1$ and $b''_{ij} = 0$ for $j \geq 1$ and $i \geq r_{M/e}(M/e) = r_M(M) - 1$. This implies that $b''_{ij} = 0$ for $j \geq 1$ and $i \geq r_M(M)$ and thus we get $b_{ij} = 0$.
Now let $i > r_M(M)$ and $j = 0$. By the definition of the Tutte polynomial the highest power of $(x - 1)$ and thus x is $r_M(M)$, which equals $cr_M(\emptyset)$. Thus all coefficients of $T(M; x, y)$ equal 0 if $i > r_M(M)$.
For the next case, let $i = r(M)$ and $j = 0$. Then it is with the previous case

and Proposition 6

$$\begin{aligned} b_{r_M(M),0} &= b'_{r_M(M),0} + b''_{r_M(M),0} \\ &= \underbrace{b'_{r_{M\setminus e}(M\setminus e),0}}_{=1 \text{ by ind. hyp.}} + \underbrace{b''_{r_{M/e}(M/e)+1,0}}_{=0} \\ &= 1. \end{aligned}$$

(ii) For proving part (ii) we proceed in the same way as in part (i).
Considering the case $i \geq 1$ and $j \geq n(M)$ we get by induction hypothesis and Corollary 4 that $b'_{ij} = 0$ for $i \geq 1$ and $j \geq n_{M\setminus e}(M \setminus e) = n_M(M) - 1$. Moreover it follows that $b''_{ij} = 0$ for $i \geq 1$ and $j \geq n_{M/e}(M/e) = n_M(M)$. That implies that $b_{ij} = 0$ for $i \geq 1$ and $j \geq n_M(M)$.
Now let $i = 0$ and $j > n_M(M)$. By the definition of the Tutte polynomial the highest power of $(y-1)$ and thus y is $n_M(M)$. Thus all coefficients of $T(M; x, y)$ equal 0 if $i > n_M(M)$.
For the last case, let $i = 0$ and $j = n_M(M)$. Then it is with the previous case and Corollary 4

$$\begin{aligned} b_{0,n_M(M)} &= b'_{0,n_M(M)} + b''_{0,n_M(M)} \\ &= \underbrace{b'_{0,n_{M\setminus e}(M\setminus e)+1}}_{=0} + \underbrace{b''_{0,n_{M/e}(M/e)}}_{=1 \text{ by ind. hyp.}} \\ &= 1. \end{aligned}$$

□

Lemma 3 (stated in [2, Lemma 6.2.]). *Let $M = (E, \mathcal{I}(M))$ be a matroid with $r_M(M) = n$ and let $e \in E$ be a nonfactor.*
If f_n^{kj} counts the flats of corank k and nullity j in a matroid of rank n, then

$$f_n^{kj}(M) = f_n^{kj}(M \setminus e) + f_{n-1}^{kj}(M/e) - \bar{f}_{n-1}^{k,j+1}(M/e), \tag{1}$$

where $\bar{f}$ counts only those closed sets of M/e, which are not closed in M.
In particular, if e is a nonfactor of all flats of corank k and nullity $j+1$ in M, then $\bar{f}_{n-1}^{k,j+1}(M/e) = f_{n-1}^{k,j+1}(M/e)$.

Proof (adapted from [2, Proof 6.2.]). First, we consider the *in particular*-part of the Lemma. We show that a set $F \subseteq E \setminus e$ cannot exist which is both closed in M/e and in M if e is a nonfactor of F in M. By Lemma 1, F is a closed set in M/e if and only if $F \cup e$ is closed in M. Assume that F is a flat in M, then $r_M(F \cup e) > r_M(F)$. This statement is true if and only if e is in every basis of $F \cup e$, which means that e is an isthmus of $F \cup e$. That is a contradiction to the assumption that e is a nonfactor of all flats of M.

We denote equation (1) by $(a) = (b) + (c) - (d)$ and consider the contributions of the different parts subsequently.

Flats of M of corank k and nullity j either contain e or do not contain e. The number of flats of M containing e is equal (c) by Lemma 1.

We now consider the flats $F \subseteq E \setminus e$ of $M \setminus e$ of corank k and nullity j that are counted in (b). By Proposition 6.(1) it is

$$\begin{aligned} F = cl_{M\setminus e}(F) &= \{x \in E \setminus e : r_{M\setminus e}(F \cup x) = r_{M\setminus e}(F)\} \\ &= \{x \in E \setminus e : r_M(F \cup x) = r_M(F)\} \\ &= cl_M(F) \setminus e. \end{aligned}$$

On the one hand, the flats of $M \setminus e$ counted in (b) are flats of M that do not contain e and thus are counted in (a) since $F = cl_M(F) \setminus e = cl_M(F)$. On the other hand, the other flats of $M \setminus e$ counted in (b) are those subsets $F \subseteq E \setminus e$ such that $e \in cl_M(F)$. This means that $F \cup e$ is closed in M but F is not, which follows from the equation above. Particularly, the closure of F in M is $F \cup e$ and hence e is not an isthmus of $F \cup e$. With the different parts of Proposition 6 it is

$$\begin{aligned} k = cr_{M\setminus e}(F) &= r_{M\setminus e}(M \setminus e) - r_{M\setminus e}(F) \\ &\overset{(1),(3)}{=} r_M(M) - r_M(F) \\ &= r_M(M) - r_M(F \cup e) \\ &\overset{(2),(4)}{=} r_{M/e}(M/e) + 1 - (r_{M/e}(F) + 1) \\ &= r_{M/e}(M/e) - r_{M/e}(F) \\ &= cr_{M/e}(F), \end{aligned}$$

and with $r_{M/e}(F) = r_M(F) - 1 = r_{M\setminus e}(F) - 1$ as in the equation above it is

$$\begin{aligned} n_{M/e}(F) &= |F| - r_{M/e}(F) \\ &= |F| - r_{M\backslash e}(F) + 1 \\ &= j + 1. \end{aligned}$$

These subsets F with corank k and nullity $j+1$ in M/e are exactly those sets counted in (d). □

Theorem 3 (stated in [2, Theorem 6.3.]). *Let $M = (E, \mathcal{I}(M))$ be a matroid of rank n with Tutte polynomial $T(M; x, y) = \sum_{i,j} b_{ij} x^i y^j$. If all p-element subsets of E are independent, then the number of flats of M of corank k and nullity j is counted for all j and for $k \geq n - p$ or $k = 0$ by*

$$f_n^{kj}(M) = \sum_{s=k}^{n} \sum_{t=0}^{n-s} (-1)^t \binom{n-s}{t} \binom{s}{k} b_{s,j+t}. \tag{2}$$

In particular, this gives for $k > n - p$: $f_n^{kj}(M) = \delta(0, j)\binom{n}{k}$.

Proof (based on [2, Proof 6.3.]). Let $i \in \{1, \ldots, p\}$. For $k = n - (p - i)$ all subsets of E with corank k and thus rank $p - i$ have nullity 0, since all $(p - i)$-element subsets of E are independent by assumption. Furthermore, all subsets with cardinality $p - i$ for $1 \leq i \leq p$ are closed in M, since all $p - i + 1$ subsets are independent in M. This means that there are no flats F of M with $cr_M(F) = k > n - p$ and $n_M(F) \neq 0$.
Hence the number of flats with corank k, which means rank and cardinality equals $n - k$, is $\binom{n}{n-k}$. All in all we get $f_n^{kj}(M) = \delta(0, j)\binom{n}{k}$ if $k > n - p$.
To show that equation (2) holds for all $k \geq n - p$ and $k = 0$, we use induction on the number of nonfactors of M.
We consider the matroid with zero nonfactors, the pre-Boolean algebra B_{nm}. It is $r(B_{nm}) = n$ and $n(B_{nm}) = |B_{nm}| - n = m$. A flat of B_{nm} is a subset $F = B_{n'm}$ with $n' \leq n$. Its nullity is $n(F) = n' + m - n' = m$. Thus all flats of B_{nm} have nullity m. With $k = cr(F) = r(B_{nm}) - r(B_{n'm}) = n - n'$ it is $f_n^{kj} = \delta(j, m)\binom{n}{n'} = \delta(j, m)\binom{n}{k}$. Considering the Tutte polynomial $T(B_{nm}; x, y) = x^n y^m$ we see that $b_{s,j+t} \neq 0$ if and only if $s = n$ and $t = m - j$. In this case we get

$$(-1)^0 \binom{0}{m-j} \binom{n}{k} = \binom{n}{k} \delta(j, m) = f_n^{kj}(B_{nm})$$

and the base clause is proved.

Before conducting the induction step, we first look at the size of the independent sets of $M \setminus e$ and M/e to extend the induction hypothesis to the latter.
By Corollary 5 all p-element subsets of E are independent in M if and only if all p-element subsets of $E \setminus e$ are independent in $M \setminus e$ and all $(p-1)$-element subsets of $E \setminus e$ are independent in M/e. Therefore, we obtain for $M \setminus e$ that $k \geq n - p$ and for M/e that $k \geq (n-1) - (p-1) = n - p$. Hereby is k the difference of the rank of the matroid and ρ, where all ρ-element subsets of the matroid are independent.
We now assume that (2) is proved for matroids with less than $q > 0$ nonfactors and consider M with q nonfactors including e. Remark that if M has q nonfactors then $M \setminus e$ and M/e have less than q nonfactors and we can assume that the induction hypothesis (2) holds for $M \setminus e$ and M/e. In the following, we will use the recursion (1) with $\bar{f}$ replaced with f. For this reason we show:

Claim:
All closed sets with corank k and nullity $j+1$ of M/e are not closed in M and thus $f = \bar{f}$.

Proof of the claim:
By Lemma 1, $F \subseteq E \setminus e$ is a flat in M/e if and only if $F \cup e$ is a flat in M. But since e is a nonfactor of M, e is not an isthmus of the flat $F \cup e$ in M and $r_M(F \cup e) = r_M(F)$, which means that F is not closed in M.□

With $T(M \setminus e; x, y) = \sum b'_{ij} x^i y^j$ and $T(M/e; x, y) = \sum b''_{ij} x^i y^j$ we get with (1) and the Deletion-Contraction Formula at $(*)$:

$$\begin{aligned}
f_n^{kj}(M) &= f_n^{kj}(M \setminus e) + f_{n-1}^{kj}(M/e) - f_{n-1}^{k,j+1}(M/e) \\
&= f_n^{kj}(M \setminus e) + \sum_{s=k}^{n-1} \sum_{t=0}^{n-s} \left[(-1)^t \binom{n-s-1}{t} \binom{s}{k} b''_{s,j+t} - (-1)^{t-1} \binom{n-s-1}{t-1} \binom{s}{k} b''_{s,j+t} \right] \\
&= \sum_{s=k}^{n} \sum_{t=0}^{n-s} (-1)^t \binom{n-s}{t} \binom{s}{k} b'_{s,j+t} + \sum_{s=k}^{n-1} \sum_{t=0}^{n-s} (-1)^t \binom{n-s}{t} \binom{s}{k} b''_{s,j+t} \\
&= \sum_{s=k}^{n} \sum_{t=0}^{n-s} (-1)^t \binom{n-s}{t} \binom{s}{k} \left(b'_{s,j+t} + b''_{s,j+t} \right) - \binom{n}{k} b''_{nj} \\
&\overset{(*)}{=} \sum_{s=k}^{n} \sum_{t=0}^{n-s} (-1)^t \binom{n-s}{t} \binom{s}{k} b_{s,j+t},
\end{aligned}$$

since $b''_{nj} = 0$ by the properties of the Tutte coefficients and $r_{M/e}(M/e) = n - 1$. □

Theorem 4 (stated in [2, Theorem 6.6.]). *If $M = (E, \mathcal{I})$ is a matroid with $|E| \geq n$ and Tutte polynomial $T(M; x, y) = \sum_{ij} b_{ij} x^i y^j$, then the following identity holds among the coefficients b_{ij}:*

$$I_n(M) = \sum_{s=0}^{n-1} \sum_{t=0}^{n-s-1} (-1)^t \binom{n-s-1}{t} b_{st} = 0.$$

Proof (adapted from [2, Proof 6.6.]). It suffices to show that $I_n(M) = 0$ for all M with $|E| = n$. For $|E| = k \geq n$ we first consider the pre-Boolean algebra B_{ij} with $i + j \geq n$. The coefficients involved in I_n are those b_{st} such that $n - s - 1 \geq t$, which is equivalent to $s + t < n$. The only coefficient unequal zero in $T(M; x, y)$ is b_{ij}, but $i + j \geq n$. Hence $I_n(B_{ij}) = 0$ holds for all pre-Boolean algebras B_{ij} with $i + j \geq n$ elements. The base clause is proven for a matroid with zero nonfactors. Now consider a matroid M with $|E| = k + 1 > n$, that is not a pre-Boolean algebra. Thus it contains a nonfactor $e \in E$. By the Deletion-Contraction Formula, it is $T(M; x, y) = T(M \setminus e; x, y) + T(M/e; x, y)$ with $|M \setminus e| = |M/e| = |E| - 1$. Hence the induction hypothesis holds for the two latter polynomials and we get $I_n(M) = I_n(M \setminus e) + I_n(M/e) = 0 + 0 = 0$.
We now consider a matroid M on a n-element ground set and assume that $r(M) = n'$ and $n(M) = m'$ with $n' + m' = n$. If $m' = 0$, then $M = B_n$ and $I_n(B_n) = 0$ because the only coefficients involved in $I_n(B_n)$ are equal to zero. So we assume $m' > 0$. The only flat of M of corank 0 is M and its nullity equals m'. Hence $f_n^{0,i}(M) = 0$ for $1 \leq i \leq m' - 1$. This implies

$$\begin{aligned}
0 &= \sum_{i=0}^{m'-1} \binom{m'-1}{i} f_n^{0,i} \\
&\overset{(2)}{=} \sum_{s=0}^{n'} \sum_{i=0}^{m'-1} \sum_{t=0}^{n'-s} (-1)^{i+t} \binom{m'-1}{i} \binom{n'-s}{t} b_{s,i+t} \\
&\overset{j:=i+t}{=} \sum_{s=0}^{n'} \sum_{j=0}^{n-s-1} (-1)^j \sum_{t=0}^{j} \binom{m'-1}{j-t} \binom{n'-s}{t} b_{sj} \\
&\overset{(*)}{=} \sum_{s=0}^{n'} \sum_{j=0}^{n-s-1} (-1)^j \binom{n-s-1}{j} b_{sj} \\
&= \sum_{s=0}^{n-1} \sum_{j=0}^{n-s-1} (-1)^j \binom{n-s-1}{j} b_{sj} \\
&= I_n(M), \text{ since } b_{sj} = 0 \text{ for all } s > n'.
\end{aligned}$$

The equality $(*)$ holds by *Vandermonde's identity*

$$\sum_{t=0}^{j} \binom{a}{j-t}\binom{b}{t} = \binom{a+b}{j}$$

with $a = m' - 1, b = n' - 1$ and hence $a + b = m' + n' - s + 1 = n - s - 1$. □

Corollary 6 (stated in [4, Lemma 1]). *Let M be a matroid with $|E| = m$ and Tutte polynomial $T(M; x, y) = \sum b_{ij} x^i y^j$. Then*

$$\sum_{s=0}^{k} \sum_{t=0}^{k-s} (-1)^t \binom{k-s}{t} b_{st} = 0 \qquad (B_k)$$

holds for $k \in \{0, 1, \ldots, m-1\}$.

Proof. We obtain the equations (B_k) by setting $k := n - 1$ in the identity $I_n(M)$ in Theorem 4. □

5 Proof of Brylawski's Conjecture

This section is adapted from [4].

Let $M = (E, \mathcal{I})$ be a connected matroid with $|E| = m$ and let $T(M; x, y) = \sum b_{ij} x^i y^j$ be its Tutte polynomial. In order to obtain a contradiction, we assume that there is a non-trivial factorization

$$T(M; x, y) = A(x, y)C(x, y) \qquad (3)$$

with $A(x, y) = \sum a_{ij} x^i y^j$ and $C(x, y) = \sum c_{ij} x^i y^j$.

Since $b_{00} = 0$ by Lemma 2.(1), either $a_{00} = 0$ or $c_{00} = 0$. We assume that $a_{00} = 0$. According to Lemma 2.(2) it is $b_{10} > 0$, so we get $0 < b_{10} = a_{00}c_{10} + a_{10}c_{00} = a_{10}c_{00}$ and thus by assumption $c_{00} \neq 0$. In the following we will prove that $c_{00} = 0$ to obtain the desired contradiction.

Definition. Let $P(x, y) = \sum p_{ij} x^i y^j$, where $P(x, y)$ does not consist of mixed terms only. Then we define

$$r_P(x) := \max\{i : p_{i0} \neq 0\}, \qquad r_P(y) := \max\{j : p_{0j} \neq 0\}$$

and

$$m(P) := r_P(x) + r_P(y).$$

Thus $r_P(x)$ and $r_P(y)$ denote the highest power of the terms x and y, respectively.
As M is connected it follows from Lemma 2.(3) that neither x nor y are factors of $T(M; x, y)$. Therefore they are not factors of $A(x, y)$ and $C(x, y)$ neither and we may apply the previous definitions to A and C.
By Lemma 2.(4) it follows that $r_T(x) = r_M(M)$ and $r_T(y) = n_M(M)$ and thus with (3) it is $m = m(T) = m(A) + m(C)$. Since we are assuming a non-trivial factorization of $T(M; x, y)$ we get that $r_A(x), r_A(y) \leq m(A) < m$.

Lemma 4. *Let $A(x, y)$ be the polynomial in* (3)*. Then the following holds:*
If $i \geq r_A(x)$ or $j \geq r_A(y)$, then $a_{ij} = 0$ except if $i = r_A(x)$ and $j = 0$ or $i = 0$ and $j = r_A(y)$.

Remark. The above-mentioned property for the polynomial $A(x, y)$ is a property, which holds for a Tutte polynomial. This was proven in Lemma 2.(4).

Proof. Let $\alpha := \max\{i : a_{ij} \neq 0 \text{ for some } j\}$ and $\beta := \max\{j : a_{\alpha j} \neq 0\}$. Analogously let α' and β' be defined for the polynomial $C(x, y)$. Thus the monomials $a_{\alpha\beta}x^{\alpha}y^{\beta}$ and $c_{\alpha'\beta'}x^{\alpha'}y^{\beta'}$ are the terms of maximum degree of x in $A(x, y)$ and $C(x, y)$ respectively. Its product $a_{\alpha\beta}c_{\alpha'\beta'}x^{\alpha+\alpha'}y^{\beta+\beta'}$ is the term with maximum degree of x in $T(M; x, y)$. By Lemma 2.(4) it follows that $\alpha + \alpha' = r(M)$ and $\beta + \beta' = 0$. Since $\beta, \beta' \geq 0$ we get $\beta = 0$. This implies that $\alpha = \max\{i : a_{ij} \neq 0 \text{ for some } j\} = \max\{i : a_{i0} \neq 0\} = r_A(x)$. Thus, the maximum degree of x in $A(x, y)$ has the coefficient $a_{r_A(x),0}$.
Considering analogously $\gamma := \max\{j : a_{ij} \neq 0 \text{ for some } i\}$ and $\delta := \max\{i : a_{i\gamma} \neq 0\}$ for $A(x, y)$ and γ' and δ' for $C(x, y)$, we get that $a_{\delta\gamma}c_{\delta'\gamma'}x^{\delta+\delta'}y^{\gamma+\gamma'}$ is the term of maximum degree of y in $T(M; x, y)$. With Lemma 2.(4) it follows that $\gamma+\gamma' = n(M)$ and $\delta+\delta' = 0$. Thus $\delta = 0$ and $\gamma = r_A(y)$, which means that the term of maximum degree of y has the coefficient $a_{0,r_A(y)}$. □

For $k = 0, \ldots, m(A)$ we define

$$\sum_{s=0}^{k} \sum_{t=0}^{k-s} (-1)^t \binom{k-s}{t} a_{st} = 0. \tag{A_k}$$

The equation (A_k) is (B_k), only with b_{st} replaced with a_{st}, the coefficients of $A(x, y)$. Since we do not assume $A(x, y)$ being a Tutte polynomial we do not know whether the equations (A_k) hold or not.

Lemma 5. *Let $A(x, y) = \sum a_{ij}x^iy^j$ be the polynomial in* (3)*. Then there is at least one $l \in \{r_A(x), \ldots, m(A)\}$ such that the equation (A_l) does not hold.*

Proof. For $r_A(x) \le k \le m(A)$ and $i \ge 0$ let the equations (A_{ki}) be

$$\sum_{s=0}^{k} \sum_{t=0}^{k-s} (-1)^{t+i} \binom{k-s}{t} a_{s,t+i} = 0. \tag{A_{ki}}$$

Remark that the equation (A_{k0}) is the equation (A_k).

Claim:
For $i > 0$ and $r_A(x) < k \le m(A)$ the following holds for the left-hand side of the equations:

$$(A_{k-1,i}) = (A_{k,i-1}) - (A_{k-1,i-1}) \tag{4}$$

Proof of the claim:
In the following let $i > 0$ and $k > r_A(x)$.
The left-hand side of the equation $(A_{k,i-1})$ is

$$\sum_{s=0}^{k} \sum_{t=0}^{k-s} (-1)^{t+i-1} \binom{k-s}{t} a_{s,t+i-1}. \tag{5}$$

Using the fact that $\binom{k-s}{t} = \binom{k-1-s}{t} + \binom{k-1-s}{t-1}$ and assuming $\binom{a}{-b} = 0$ for $a \ge 0$ and $b > 0$ and $\binom{a}{b} = 0$ for $a < b$, (5) can be rewritten in the following way:

$$\begin{aligned}
&\sum_{s=0}^{k} \sum_{t=0}^{k-s} (-1)^{t+i-1} \binom{k-s}{t} a_{s,t+i-1} \\
=&\sum_{s=0}^{k} \sum_{t=0}^{k-s} (-1)^{t+i-1} \left[\binom{k-1-s}{t} + \binom{k-1-s}{t-1} \right] a_{s,t+i-1} + (-1)^{i-1} a_{k,i-1}
\end{aligned}$$

The last term appears because $\binom{0}{0}$ cannot be decomposed into two binomial coefficients. Since $k > r_A(x)$ we have $a_{k,i-1} = 0$ by Lemma 4. Using this it is

$$\begin{aligned}
=&\sum_{s=0}^{k-1} \sum_{t=0}^{k-1-s} (-1)^{t+i-1} \binom{k-1-s}{t} a_{s,t+i-1} + \sum_{s=0}^{k-1} \sum_{t=1}^{k-1-s} (-1)^{t-1+i} \binom{k-1-s}{t-1} a_{s,t-1+i} \\
=&\sum_{s=0}^{k-1} \sum_{t=0}^{k-1-s} (-1)^{t+i-1} \binom{k-1-s}{t} a_{s,t+i-1} + \sum_{s=0}^{k-1} \sum_{t=0}^{k-1-s} (-1)^{t+i} \binom{k-1-s}{t} a_{s,t+i}.
\end{aligned}$$

The terms of the last row are the left-hand sides of the equations $(A_{k-1,i-1})$ and $(A_{k-1,i})$, respectively. So we can write symbolically

$$(A_{k,i-1}) = (A_{k-1,i-1}) + (A_{k-1,i}),$$

which is the claimed relation.$\square$

We assume now that all equations (A_k) hold for $r_A(x) \leq k \leq m(A)$, which is leading to a contradiction. We now consider the equation $(A_{r_A(x),r_A(y)})$. Its left-hand side is

$$\begin{aligned}(A_{r_A(x),r_A(y)}) &= \sum_{s=0}^{r_A(x)} \sum_{t=0}^{r_A(x)-s} (-1)^{r_A(y)+t} \binom{r_A(x)-s}{t} a_{s,r_A(y)+t} \\ &= (-1)^{r_A(y)} \binom{r_A(x)}{0} a_{0,r_A(y)} \\ &= \pm a_{0,r_A(y)} \\ &\neq 0,\end{aligned}$$

since by Lemma 4 the only term $a_{ij} \neq 0$ involved in this equation is $a_{0,r_A(y)}$.
Otherwise, by using the recursion (4) we can express $(A_{r_A(x),r_A(y)}) \neq 0$ as the sum of equations (A_{k0}) for $r_A(x) \leq k \leq m(A) = r_A(x) + r_A(y)$. But we are assuming that these equations (A_{k0}) all equal zero. This implies $(A_{r_A(x),r_A(y)}) = 0$ and therefore we obtain a contradiction. This means that there exists a $k \in \{r_A(x), \ldots, m(A)\}$ such that $(A_{k0}) = (A_k)$ does not hold. $\square$

Lemma 6. *If the coefficients a_{ij} do not satisfy equation (A_k) for some $k \leq m(A)$ then $c_{00} = 0$.*

Proof. Let (A_k) be the first equation that does not hold. Since $k \leq m(A) < m$, the equation (B_k) holds. With the fact

$$b_{st} = \sum_{h \leq s} \sum_{l \leq t} c_{hl} a_{s-h,t-l}$$

we can rewrite the left-hand side of (B_k) in the following way:

$$
\begin{aligned}
\sum_{s=0}^{k}\sum_{t=0}^{k-s}(-1)^t\binom{k-s}{t}b_{st} =& \sum_{s=0}^{k}\sum_{t=0}^{k-s}(-1)^t\binom{k-s}{t}\sum_{h\le s}\sum_{l\le t}c_{hl}a_{s-h,t-l}\\
=& c_{00}\sum_{s=0}^{k}\sum_{t=0}^{k-s}(-1)^t\binom{k-s}{t}a_{st}\\
&+\sum_{0<h+l\le k}c_{hl}\left[\sum_{s=h}^{k}\sum_{t=l}^{k-s}(-1)^t\binom{k-s}{t}a_{s-h,t-l}\right]. \qquad (6)
\end{aligned}
$$

The coefficients of c_{hl} are similar to the left-hand side of the equations (A_k). In particular, the coefficient of c_{00} is exactly the left-hand side of (A_k). To analyze these coefficients of c_{hl} we define the equations (A'_{ni}) as follows:

$$
\sum_{s=0}^{n-i}\sum_{t=0}^{n-i-s}(-1)^{t+i}\binom{n-s}{t+i}a_{st}=0. \qquad (A'_{ni})
$$

Remark that (A'_{n0}) is the equation (A_n), which we are assuming to hold for $0\le n<k$. We will get the coefficient of c_{hl} in (6), if we change the indices $s+h\to s$ and $t+l\to t$ in the left-hand side of the equation $(A'_{k-h,l})$. Moreover, the binomial $\binom{k-s}{t}=0$ for $s>k-l$ and $t\ge l$. So we can write symbolically

$$
(B_k)=c_{00}(A_k)+\sum_{0<h+l\le k}c_{hl}(A'_{k-h,l}). \qquad (7)
$$

Claim:
The equation (A'_{ni}) holds for $1\le n\le k$ and $1\le i\le n$.

Proof of the claim:
We use induction on n. If $n=1$ then $i=1$ is the only possible value for i and (A'_{11}) reduces to $a_{00}=0$, which was supposed from the beginning. We now assume that the induction hypothesis holds for all values less than n. By using $\binom{a}{b}=\sum_{j=1}^{a}\binom{a-j}{b-1}$ we decompose the left-hand side of the equation (A'_{ni}):

$$\sum_{s=0}^{n-i}\sum_{t=0}^{n-i-s}(-1)^{t+i}\binom{n-s}{t+i}a_{st}$$
$$=\sum_{s=0}^{n-i}\sum_{t=0}^{n-i-s}(-1)^{t+i}\sum_{j=1}^{n-s-t-i+1}\binom{n-s-j}{t+i-1}a_{st}$$
$$=\sum_{s=0}^{n-i}\sum_{t=0}^{n-i-s}\sum_{j=1}^{n-s-t-i+1}(-1)^{t+i}\binom{n-s-j}{t+i-1}a_{st}$$
$$=-\sum_{j=1}^{n-i+1}\sum_{s=0}^{(n-j)-(i-1)}\sum_{t=0}^{(n-j)-(i-1)-s}(-1)^{t+(i-1)}\binom{(n-j)-s}{t+(i-1)}a_{st}.$$

The jth term in the last sum is equal to the left-hand side of the equation $(A'_{n-j,i-1})$ for $1 \leq j \leq n-i+1$. Hence we obtain the relation

$$(A'_{ni}) = -\sum_{j=1}^{n-i+1}(A'_{n-j,i-1}).$$

For $i = 1$ the equations on the right-hand side are $(A_{n-1}), \ldots, (A_0)$, which hold since $n-1 < k$. If $i > 1$ the equations $(A'_{n-1,i-1}), \ldots, (A'_{i-1,i-1})$ hold by induction hypothesis, thus in both cases (A'_{ni}) holds.□

This result implies with equation (7) that (B_k) reduces to $c_{00}(A_k) = 0$. Since (A_k) does not hold by assumption it follows that $c_{00} = 0$. This contradiction proves this Lemma and hence Brylawski's Conjecture stated as Theorem 2. □

Remark. Actually, the proof shows that $T(M; x, y)$ is irreducible even in $\mathbb{C}[x, y]$.

6 The Arithmetic Tutte Polynomial

All following definitions can be found in [1].
Alternatively to the definition of a matroid via independent sets a matroid could be defined as follows.

Definition. A *matroid* $M = (E, r)$ is a finite set E and a *rank function* $r : 2^E \to \mathbb{N}$ satisfying

(R1) $A \subseteq E$ implies $r(A) \leq |A|$.

(R2) $A \subseteq B \subseteq E$ implies $r(A) \leq r(B)$.

(R3) $A, B \subseteq E$ implies $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$

Definition. Let $M = (E, r)$ be a matroid. If $R \subseteq S \subseteq E$ let $[R, S] := \{A : R \subseteq A \subseteq S\}$. $[R, S]$ is a *molecule* if S is the disjoint union $S = R \cup F \cup T$ for $F, T \subseteq E$ and for each $A \in [R, S]$ it is

$$r(A) = r(R) + |A \cap F|.$$

Definition. An *arithmetic matroid* $(A) = (M, m)$ is a matroid $M = (E, r)$ with a *multiplicity function* $m : 2^E \to \mathbb{N} \setminus \{0\}$ satisfying

(P) For a molecule $[R, S]$ it is

$$(-1)^{|T|} \sum_{A \in [R,S]} (-1)^{|S|-|A|} m(A) \geq 0.$$

(M1) $A \subseteq E$, $e \in E$ and $r(A \cup e) = r(A)$ implies that $m(A \cup e)$ divides $m(A)$.

(M2) $A \subseteq E$, $e \in E$ and $r(A \cup e) > r(A)$ implies that $m(A)$ divides $m(A \cup e)$.

(M3) For a molecule $[R, S]$ it is $m(R) \cdot m(S) = m(R \cup F) \cdot m(R \cup T)$.

Definition. Let $\mathcal{A}$ be an arithmetic matroid. Then its *arithmetic Tutte polynomial* is defined as

$$M_{\mathcal{A}}(x, y) = \sum_{A \subseteq E} m(A)(x - 1)^{cr(A)}(y - 1)^{n(A)}.$$

Question: Is the arithmetic Tutte polynomial of a connected arithmetic matroid irreducible in $\mathbb{Z}[x, y]$?

This thesis forms the basis of the examination of the question above.
We found out that there are only a few aspects that need to be checked. Actually, some are already proven. This is on the one hand that the Deletion-Contraction Formula also holds for arithmetic matroids (see [1]). On the other hand, there is an alternative definition of the arithmetic Tutte polynomial via local external and local internal activity of molecules in [1], which is similar to the alternative Tutte polynomial. Thus the investigation whether the *Basic Properties* (stated in Lemma 2) hold for the coefficients of the arithmetic Tutte polynomial still remains.

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