A PARTIAL ORDER ON THE SYMMETRIC GROUP AS A SUBGROUP OF THE UNITARY GROUP OVER A FINITE DIMENSIONAL COMPLEX VECTOR SPACE

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Contents

Introduction

In this thesis a partial order on the *unitary* group of a n dimensional *unitary* vector space will be defined, referring to Thomas Brady's and Colum Watt's paper "A Partial Order on the Orthogonal Group" [BW02]. In [BW02] Brady and Watt regard a finite dimensional vector space over any field and the orthogonal group with respect to a symmetric bilinear form instead.

As it is possible to find unitary matrices operating on a vector by transposing its entries, we regard the symmetric group S_n as a subgroup of the unitary group $U(\mathbb{C}^n)$.
We will prove that the partial order being transferred from $U(\mathbb{C}^n)$ to S_n is the same as the one being defined by Brady in reference paper [Bra01] using the length function of the Cayley Graph, counting the minimum amount of consecutive transpositions the permutation can be displayed as.

We find that a permutation $\tau \in S_n$ is lower than or eqal to another permutation $\sigma \in S_n$, if and only if each cycle of τ is contained in a cycle of σ, τ is ordered consistently with σ, and τ has no crossing cycles with respect to σ. Therefore we define a cycle to be ordered consistently with another permutation without using the $\leq$ relation, and we give the definition of crossing cycles not before we have characterized the meaning of a cycle c_τ being lower than or equal to a permutation σ.

We regard S_3 and S_4 as examples.

The cycle structure of a permutation σ defines a partition of the set $\{1, ..., n\}$. We will denote it $\{\sigma\}$. Since the set Π_n of partitions of the set {1,...,n} has a natural partial order, we consider the subposet of S_n given by the "allowable elements" $\mathscr{A}$, so that for each τ and $\sigma \in \mathscr{A}$ we have $\tau \leq \sigma$ if and only if $\{\tau\} \leq \{\sigma\}$. We know that the set $NCP(n) \subseteq \Pi_n$ of non crossing partitions is a partial ordered set and with a meet and join operation it does form a lattice. Since for $h : \mathscr{A} \to \Pi_n, \sigma \mapsto \{\sigma\}$ it is $im(h) = NCP(n)$, we can define meet and join for elements of $\mathscr{A}$ with regard to the definition for $NCP(n)$. Finally we prove that the set of allowable elements $\mathscr{A}$ with the $\leq$ relation of S_n and with the given meet and join operation does form a lattice.

We generalize this statement and regard the unitary group over $\mathbb{C}^n$ again: We take a restriction of the unitary matrices to an interval $[A, C]$ with $A, C \in U(\mathbb{C}^n)$, $A \leq C$, and if we define $m := |V_C| - |V_A|$, we find that $[A, C]$ is isomorphic to the lattice of subspaces of $\mathbb{C}^m$. Using this at the end we get to a strong version of the Cartan-Dieudonné Theorem, with respect to the fact that $char(\mathbb{C}) \neq 2$.

1 A Partial Order on the Unitary Group

Let $n \in \mathbb{N}$ be a finite number and regard $\mathbb{C}^n$ the n-dimensional complex vector space. The complex scalar product of two complex vectors

$$a := \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \text{ and } b := \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \text{ is defined via } < a, b >:= \sum_{i=1}^{n} a_i \bar{b_i}.$$

The set of matrices $\{M \in \mathbb{C}^{n \times n} \big| < Me_i, Me_i >= 1 \ \forall i \in \{1, ..., n\}$ and $< Me_i, Me_j >= 0 \ \forall i, j \in \{1, ..., n\},\ i \neq j\}$ represents the *unitary group* $U(\mathbb{C}^n)$ together with the matrix multiplication as its binary operation.

For all $A \in U(\mathbb{C}^n)$ we define $W_A := ker(A - I)$ and $V_A := im(A - I)$. We will write $|W_A|$ for $dim(W_A)$ and $|V_A|$ for $dim(V_A)$.

For any subspaces $X, Y \subseteq \mathbb{C}^n$ we will write $X \perp Y$ for the unitary direct sum of X and Y and we define $X^{\perp} := \{y \in \mathbb{C}^n \big| \forall x \in X :< x, y >= 0\}$ as the subspace of all vectors being unitary to X.

Proposition 1.1 (Brady and Watt, Reference [BW02], Proposition 1)
For all $A \in U(\mathbb{C}^n)$ we get the following states:

1. W_A is the +1-eigenspace of $f_A : \mathbb{C}^n \to \mathbb{C}^n,\ x \mapsto Ax$.
2. $\mathbb{C}^n = V_A \perp W_A$.

Proof.

1. $x \in W_A \Leftrightarrow x \in ker(A - I) \Leftrightarrow (A - I)x = 0 \Leftrightarrow Ax = x \Leftrightarrow x \in E_{f_A}(1)$, where $E_{f_A}(1) := \{x \in \mathbb{C}^n \big| f_A(x) = 1 \cdot x\}$ is the +1-eigenspace of f_A.
2. This can be proofed analoguely to reference [BW02] with the complex scalar product instead of the real scalarproduct:
   - The subspaces are unitary: Regard any $v \in V_A$, $w \in W_A$. Since $V_A = im(A - I)$ there exists $z \in \mathbb{C}^n$ so that $(A - I)z = v$. We get

     $$\begin{aligned} < v, w > &= < v, (A - I)z >=< v, Az - z >=< v, Az > - < v, z > \\ &= < Av, Az > - < v, z >= 0 \end{aligned}$$

     since $A \in U(\mathbb{C}^n) \Leftrightarrow < Ax, Ay >=< x, y > \forall x, y \in \mathbb{C}^n$.
   - Since the subspaces are unitary and the dimensions are complementary we get $\mathbb{C}^n = V_A \oplus W_A$.

$\square$

Corollary 1.2

1. $V_A^\perp = W_A$ *and* $W_A^\perp = V_A$
2. $|V_A| = n - |W_A|$ *and* $|W_A| = n - |V_A|$
3. $W_A \cap W_B \subseteq W_{AB}$

Proof. The first and the second state follow directly from Proposition 1.1.2. Regard $x \in W_A \cap W_B$. Using Proposition 1.1.1. we get for all $x \in W_A \cap W_B$: $Ax = x$ and $Bx = x$. It follows $(AB)x = A(Bx) = Ax = x$, which means $x \in W_{AB}$.

$\square$

Lemma 1.3 *For any Subspaces* $X, Y \subseteq \mathbb{C}^n$*:*

1. $(X+Y)^\perp = X^\perp \cap Y^\perp$ *and* $X^\perp + Y^\perp = (X \cap Y)^\perp$
2. $|X| + |Y| = |X+Y| + |X \cap Y|$

Proof.

1. For any $z \in \mathbb{C}^n$ we have $z \in (X+Y)^\perp \Leftrightarrow \forall s \in X+Y :< z, s >= 0 \Leftrightarrow \forall x \in X\ \forall y \in Y :< z, x+y >= 0 \Leftrightarrow \forall x \in X\ \forall y \in Y :< z, x > + < z, y >= 0$. Because $0 \in X$ and $0 \in Y$ we get, that this is equivalent to $\forall x \in X :< z, x >= 0 \wedge \forall y \in Y :< z, y >= 0$, which exactly means $z \in X^\perp \cap Y^\perp$. The other state follows directly by regarding $X^\perp$ instead of X and $Y^\perp$ instead of Y and using $(X^\perp)^\perp = X$ and $(Y^\perp)^\perp = Y$. It is $(X^\perp)^\perp = X$, because $\mathbb{C}^n = X \oplus X^\perp = (X^\perp)^\perp \oplus X^\perp$ ([Fis79], "Orthonormalisierungssatz", Corollary 2).
2. A proof for this common state can be found e.g. in reference [Fis79] ("Dimensionsformel für Summen").

$\square$

Proposition 1.4 *For all* $A, B \in U(\mathbb{C}^n)$*:* $|V_{AB}| \leq |V_A| + |V_B|$.

Proof. Regard

$$\begin{aligned}
|V_A| + |V_B| &\underset{\substack{\uparrow \\ \text{Corollary 1.2.2}}}{=} n - |W_A| + n - |W_B| = 2n - (|W_A| + |W_B|) \\
&\underset{\substack{\uparrow \\ \text{Lemma 1.3.2}}}{=} 2n - (|\underbrace{W_A + W_B}_{\subseteq \mathbb{C}^n}| + |\underbrace{W_A \cap W_B}_{\substack{\subseteq W_{AB} \\ \text{Corollary 1.2.3}}}|) \\
&\geq 2n - (n + |W_{AB}|) = n - |W_{AB}| \underset{\substack{\uparrow \\ \text{Corollary 1.2.2}}}{=} |V_{AB}|
\end{aligned}$$

□

Definition 1.5 (Brady and Watt, Reference [BW02], Definition 1) For all $A \in U(\mathbb{C}^n)$ and for all $B \in U(\mathbb{C}^n)$ we define

$$A \leq C :\Leftrightarrow |V_C| = |V_A| + |V_{A^{-1}C}|$$

Proposition 1.6 (Brady and Watt, Reference [BW02], Proposition 3)
The relation $\leq$ is a partial order on $U(\mathbb{C}^n)$.

Proof.

- Reflexivity: It is $|V_I| = |im(I - I)| = 0 \Leftrightarrow$ for all $A \in U(\mathbb{C}^n)$:
$|V_A| = |V_A| + |V_I| = |V_A| + |V_{A^{-1}A}| \Leftrightarrow \forall A \in U(\mathbb{C}^n) : A \leq A$.
- Antisymmetry: Suppose $A \leq C$ and $C \leq A$.
Then $|V_C| \underset{\substack{\uparrow \\ A \leq C}}{=} |V_A| + |V_{A^{-1}C}| \underset{\substack{\uparrow \\ C \leq A}}{=} |V_C| + |V_{C^{-1}A}| + |V_{A^{-1}C}|$.
This gives $V_{C^{-1}A} = V_{A^{-1}C} = \{0\}$ and $A = C$.
- Transitivity: Suppose $A \leq B$ and $B \leq C$. Then

$$\begin{aligned} |V_C| = |V_{AA^{-1}C}| &\underset{\substack{\uparrow \\ \text{Proposition 1.4}}}{\leq} |V_A| + |V_{A^{-1}C}| \\ &= |V_A| + |V_{A^{-1}BB^{-1}C}| \\ &\underset{\substack{\uparrow \\ \text{Proposition 1.4}}}{\leq} |V_A| + |V_{A^{-1}B}| + |V_{B^{-1}C}| \\ &\underset{\substack{\uparrow \\ A \leq B \wedge B \leq C}}{=} |V_A| + (|V_B| - |V_A|) + (|V_C| - |V_B|) = |V_C|. \end{aligned}$$

So we have actually equality for every line, which gives $|V_C| = |V_A| + |V_{A^{-1}C}|$ and further $A \leq C$.

□

Corollary 1.7 (Brady and Watt, Reference [BW02], Corollary 1)
For all $A \in U(\mathbb{C}^n)$, for all $C \in U(\mathbb{C}^n)$:

$$A \leq C \Leftrightarrow V_C = V_A \oplus V_{A^{-1}C}$$

Proof. Define $B := A^{-1}C \in U(\mathbb{C}^n)$. With regard to reference [BW02], Corollary 1, we show $|V_{AB}| = |V_A| + |V_B| \Leftrightarrow V_{AB} = V_A \oplus V_B$.
The proof of Proposition 1.4 gives $|V_A| + |V_B| \geq |V_{AB}|$ and further

$$|V_{AB}| = |V_A| + |V_B| \quad \Leftrightarrow \quad \begin{cases} W_A + W_B = \mathbb{C}^n \\ W_A \cap W_B = W_{AB} \end{cases}$$

Using Lemma 1.3.1 and Corollary 1.2.1 we get that this is equivalent to

$$\Leftrightarrow \quad \begin{cases} V_A \cap V_B = \{0\} \\ V_A + V_B = V_{AB} \end{cases} \quad \Leftrightarrow V_A \oplus V_B = V_{AB}$$

□

Lemma 1.8 *For all $A, B \in U(\mathbb{C}^n)$: $A^{-1}B \in U(\mathbb{C}^n), BA^{-1} \in U(\mathbb{C}^n)$ and*

$$\begin{aligned} A \leq B \quad &\Leftrightarrow \quad A^{-1}B \leq B \\ &\Leftrightarrow \quad BA^{-1} \leq B. \end{aligned}$$

Proof. The states $A^{-1}B \in U(\mathbb{C}^n)$ and $BA^{-1} \in U(\mathbb{C}^n)$ are trivial since $U(\mathbb{C}^n)$ is a group.

Since B is invertible, we get $|V_{B^{-1}AB}| = |im((B^{-1}AB) - I)|$
$= |im(B^{-1}(A - I)B)| = |im(A - I)| = |V_A|$. Using this, we get

$$\begin{aligned} A \leq B \Leftrightarrow |V_B| \quad &= \quad |V_{A^{-1}B}| + |V_A| \\ &= \quad |V_{A^{-1}B}| + |V_{B^{-1}AB}| \\ &= \quad |V_{A^{-1}B}| + |V_{(A^{-1}B)^{-1}B}| \\ \Leftrightarrow A^{-1}B \leq B & \end{aligned}$$

and

$$\begin{aligned} A \leq B \Leftrightarrow |V_B| \quad &= \quad |V_{A^{-1}B}| + |V_A| \\ &= \quad |V_{B(A^{-1}B)B^{-1}}| + |V_{AB^{-1}B}| \\ &= \quad |V_{BA^{-1}}| + |V_{(BA^{-1})^{-1}B}| \\ \Leftrightarrow BA^{-1} \leq B. & \end{aligned}$$

□

Lemma 1.9 *For all $A, B, C \in U(\mathbb{C}^n)$: If $A \leq B \leq C$, then*
$A^{-1}B \leq A^{-1}C$ and $B^{-1}C \leq A^{-1}C$.

Proof. (Brady, paper [Bra01], Lemma 3.10) Assume $A \leq B \leq C$. Then:

$$\begin{aligned} A \leq B \quad &\Leftrightarrow \quad |V_B| = |V_A| + |V_{A^{-1}B}| \quad (1) \\ B \leq C \quad &\Leftrightarrow \quad |V_C| = |V_B| + |V_{B^{-1}C}| \quad (2) \\ \text{With transitivity: } A \leq C \quad &\Leftrightarrow \quad |V_C| = |V_A| + |V_{A^{-1}C}| \quad (3) \end{aligned}$$

Inserting (1) into (2) gives $|V_C| = |V_A| + |V_{A^{-1}B}| + |V_{B^{-1}C}|$. Regarding the difference with (3) we get

$$\begin{aligned} & 0 = |V_{A^{-1}B}| + |V_{B^{-1}C}| - |V_{A^{-1}C}| \\ \Leftrightarrow \quad & |V_{A^{-1}C}| = |V_{A^{-1}B}| + |V_{(A^{-1}B)^{-1}A^{-1}C}| \\ \Leftrightarrow \quad & A^{-1}B \leq A^{-1}C \end{aligned}$$

which is the first result. Using Lemma 1.8 for $\tilde{A} := A^{-1}B \in U(\mathbb{C}^n)$ and $\tilde{B} := A^{-1}C \in \mathbb{C}^n$ we get

$$\begin{aligned} \tilde{A}^{-1}\tilde{B} &\leq \tilde{B} \\ \Leftrightarrow B^{-1}AA^{-1}C &\leq A^{-1}C \\ \Leftrightarrow B^{-1}C &\leq A^{-1}C \end{aligned}$$

which is the second result.

□

Note: In reference (Brady, paper [Bra01], Lemma 3.9 and Lemma 3.10) one direction of Lemma 1.8 and Lemma 1.9 are given for the special case of the symmetric group. One part of Lemma 1.9 is stated in (Brady and Watt, reference [BW02], Proposition 3) for the partial order on the orthogonal group.

Theorem 1.12 (Brady and Watt, Reference [BW02], Theorem 1) *We fix $C \in U(\mathbb{C}^n)$. For all subspaces $S \subseteq V_C$ there exists a unique $A \in U(\mathbb{C}^n)$ so that $A \leq C$ and $V_A = S$.*

In order to prepare the proof for this theorem, we define the subspace $X_S := \{x \in V_C \big| (C-I)x \in S\}$. Since $S \subseteq V_C$ we get $(C-I)X_S = S$. We now get the following Lemmata:

Lemma 1.10 (Brady and Watt, Reference [BW02], Lemma 1)

$$X_S \oplus S^{\perp} = \mathbb{C}^n$$

Proof.

1. Since $(C-I)$ is invertible when restricted to $V_C = im(C-I)$, we get a bijection $X_S \to S, x \mapsto (C-I)x$. This leads to $|X_S| = |S|$.

2. $X_S \cap S^{\perp} = \{0\}$: Regard any vector in the intersection $x \in X_S \cap S^{\perp}$. We find some $s \in S$ such that $Cx = x + s$. Using $C \in U(\mathbb{C}^n)$ we now get:

   $$< x, x > = < Cx, Cx > = < x + s, x + s > = < x, x > + < s, s >$$

   which leads to $< s, s > = 0$, and further $s = 0$. So $(C-I)x = 0$. Since $(C-I)$ is an isomorphism when restricted to X_S, we get $x = 0$.

Using 1. we get $|S^{\perp}| = n - |S| = n - |X_S|$ and together with 2. we get $X_S \oplus S^{\perp} = \mathbb{C}^n$.

□

Lemma 1.11 (Brady and Watt, Reference [BW02], Lemma 2 and Lemma 3)
If $A \leq C$ *and* $V_A = S$*, then* $W_{A^{-1}C} = W_C \bot X_S$*.*

Proof.

1. $W_C \bot X_S$: This follows since $W_C \bot V_C$ and $X_S \subseteq V_C$.

2. $W_{A^{-1}C} \subseteq W_C \bot X_S$: Regard any $x \in W_{A^{-1}C}$. Then $A^{-1}Cx = 0$ and $Cx = Ax$ and $(C - I)x = (A - I)x$. Since $\mathbb{C}^n = W_C \bot V_C$ we find a unique $y \in W_C$ and a unique $z \in V_C$ such that $x = y + z$. We get $z \in X_S$ because of $(C - I)z = (C - I)x = (A - I)x \in V_A = S$.

3. $|W_{A^{-1}C}| = |W_C + X_S|$: We use $A \leq C \Leftrightarrow |V_{A^{-1}C}| = |V_C| - |V_A|$ and $|V_A| = |X_S|$ and get $|W_{A^{-1}C}| = n - |V_{A^{-1}C}| = n - (|V_C| - |V_A|) = n - |V_C| + |X_S| = |W_C| + |X_S|$. Because of $W_C \cap V_C = \emptyset$ and $X_S \subseteq V_C$ we get $W_C \cap X_S = \emptyset$ and with Lemma 1.3.2 it is $|W_C| + |X_S| = |W_C + X_S|$.

(1.), (2.) and (3.) together give the result.

□

Proof of Theorem 1.12 Because of Lemma 1.10 we can define a matrix P describing a projection on X_S, via $Px = x$ for any $x \in X_S$, and $Ps = 0$ for any $s \in S^{\bot}$.

According to Brady and Watt (Reference [BW02], Lemma 4) we now define $A := I + (C - I)P$, and show:

- $V_A = S$: Because of the definitions of A, P and X_S we get $V_A = im(A - I) = im((C - I)P) = (C - I)X_S = S$.

- $A \leq C$: For any $x \in \mathbb{C}^n$ we have for any $B \in U(\mathbb{C}^n)$: $x \in W_B \Leftrightarrow (B - I)x = 0 \Leftrightarrow Bx = x \Leftrightarrow x = B^{-1}x \Leftrightarrow x \in W_{B^{-1}}$, and using Proposition 1.1.2 we also find $V_B = V_{B^{-1}}$. We further express:

$$\begin{aligned} C^{-1}A - I &= C^{-1}(I + (C - I)P) - I = C^{-1} + C^{-1}(C - I)P - I \\ &= C^{-1} + P - C^{-1}P - I = (C^{-1} - I) + P(I - C^{-1}) \\ &= (C^{-1} - I) - P(C^{-1} - I) = (I - P)(C^{-1} - I). \end{aligned}$$

  So we now get

$$\begin{aligned} |W_{A^{-1}C}| &= |W_{(A^{-1}C)^{-1}}| = |W_{C^{-1}A}| = |ker(C^{-1}A - I)| \\ &= |ker((I - P)(C^{-1} - I))| \end{aligned}$$

  Since $C^{-1} - I$ is an isomorhism when restricted to X_S, we further have

$$\begin{aligned} |ker((I - P)(C^{-1} - I))| &= |ker(I - P)| + |ker(C^{-1} - I)| \\ &= |X_S| + |W_C| = |S| + |W_C| = |V_A| + |W_C| \end{aligned}$$

and further

$$\begin{aligned} & |W_{A^{-1}C}| = |W_C| + |V_A| \\ \Leftrightarrow\ & n - |V_{A^{-1}C}| = n - |V_C| + |V_A| \\ \Leftrightarrow\ & |V_C| = |V_{A^{-1}C}| + |V_A| \\ \Leftrightarrow\ & A \le C \end{aligned}$$

- $A \in U(\mathbb{C}^n)$: Regard any $x, y \in \mathbb{C}^n$. Using Lemma 1.10 we can find $x = x_1 + x_2$, $y = y_1 + y_2$, with unique vectors $x_1, y_1 \in X_S$, $x_2, y_2 \in S^\perp$. Since the definition of A leads to the identities $Ax_1 = Cx_1$, $Ax_2 = x_2$, $Ay_1 = Cy_1$ and $Ay_2 = y_2$, we can show $< Ax, Ay >=< x, y >$ in the same way as Brady and Watt (Reference [BW02], Lemma 5).
- A is unique: (Brady, Watt, Reference [BW02], Lemma 4) Suppose there is an $A' \le C$ and $V_{A'} = S$. Then $W_{A'} = V_{A'}^\perp = S^\perp \Rightarrow (A - I)s = 0\ \forall s \in S^\perp$. On the other hand Lemma 1.11 gives $X_S \subseteq W_{A'^{-1}C}$, such that for all $x \in X_S$ we have $A'^{-1}Cx = 0$ which means $A'x = Cx$. Since $(A' - I)s = 0$ for all $s \in S^\perp$ and $(A' - I)x = (C - I)x$ for all $x \in X_S$, we get $A' - I = (C - I)P = A - I$ and $A' = A$.

□

Corollary 1.13 (Brady and Watt, Reference [BW02], Corollary 2)
If S is an invariant subspace of C, then the unique induced transformation on S is the restriction of C to S.

Proof. We have $S \subseteq V_C$ with $(C - I)S = S \Rightarrow X_S = S$.
Then by Theorem 1.12 $A - I = (C - I)P$ with P as the projection on S: $P(s + s') = s$ for any $s \in S$ and any $s' \in S^\perp$.

□

Corollary 1.14 (Brady and Watt, Reference [BW02], Corollary 3)
If S is a one dimensional subspace of V_C, then there exists $\alpha \in \mathbb{C}$, $|\alpha| = 1, \alpha \neq 1$, such that the unitary transformation induced by C on S is given by $Az = \begin{cases} z & \text{,if } z \in S^\perp \\ \alpha z & \text{,if } z \in S. \end{cases}$

Proof. We have $S^\perp = X_S$ and $1 = |S| = |V_A| = |im(A - I)| \Rightarrow \exists \alpha' \in \mathbb{C}, \alpha' \neq 0 : (A - I)x = \alpha' x\ \forall x \in X_S \Rightarrow \exists \alpha \in \mathbb{C}, \alpha \neq 1 : Ax = \alpha x\ \forall x \in X_S$. Since $A \in U(\mathbb{C}^n)$ we have $< Ax, Ay >=< x, y >\ \forall x, y \in \mathbb{C}^n$. Especially for $x = y = e_1$ we get $1 =< e_1, e_1 >=< Ae_1, Ae_1 >=< \alpha e_1, \alpha e_1 >= \alpha\bar{\alpha} < e_1, e_1 >= |\alpha|^2$. So $|\alpha| = 1$ and $\alpha \neq 1$.

□

Note: In reference [BW02] Brady and Watt regard an n-dimensional vector space over any field $\mathbb{F}$, and the orthogonal group with respect to a symmetric bilinear form instead of the complex scalar product. Since the complex scalar product is conjugate linear and not bilinear, we can just follow $|\alpha|^2 = 1$ here, but not $\alpha^2 = 1$ like in reference [BW02]. In [BW02] we see that if $char(\mathbb{F}) \neq 2$, then it follows $\alpha^2 = 1, \alpha \neq 1 \Rightarrow \alpha = -1$, so that A would be the orthogonal reflection in $S^\perp$.

2 The Symmetric Group as a Subgroup of the Unitary Group

Lemma 2.1 *The symmetric group S_n, representing the permutations of n elements, is isomorphic to a subgroup of the unitary group of $\mathbb{C}^n$*

$$S_n < U(\mathbb{C}^n)$$

as the function

$$\begin{array}{rcl} f : S_n & \hookrightarrow & U(\mathbb{C}^n) \\ \sigma & \mapsto & f_\sigma \end{array}$$

with the unitary map

$$\begin{array}{rcl} f_\sigma : \mathbb{C}^n & \to & \mathbb{C}^n \\ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} & \mapsto & \underbrace{\begin{pmatrix} e_{\sigma(1)} \Big| & \cdots & \Big| e_{\sigma(n)} \end{pmatrix}}_{=:M_\sigma} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \end{array}$$

is a well defined monomorphism.

Proof. We can identify the linear function f_σ with the matrix M_σ. The function f is well defined, because for all $\sigma \in S_n$ the matrix M_σ is unitary. The function f is injective since distinct permutations $\tau, \sigma \in S_n$ give distinct matrices $M_\tau \neq M_\sigma$, and so $f_\tau \neq f_\sigma$.
We have for all $\tau, \sigma \in S_n$, for all $i \in \{1, ..., n\}$:

$$\begin{array}{rcl} M_\tau M_\sigma e_i & = & M_\tau e_{\sigma(i)} = e_{\tau(j)} \text{ with } j = \sigma(i) \\ & = & e_{\tau(\sigma(i))} = M_{\tau\sigma} e_i, \end{array}$$

which gives $f(\tau\sigma) = f(\tau) f(\sigma)$.

□

Proposition 2.2 We further define $W := < \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} >$ and $V := W^{\perp}$ and get

1. W is a subspace of W_{M_σ} for all $\sigma \in S_n$.
2. $W = W_{M_\sigma}$ for at least one σ.
3. For all $\sigma \in S_n$: $|W_{M_\sigma}| = 1$ if and only if $W = W_{M_\sigma}$.

Proof.

1. $W \leq W_{M_\sigma}$ as for any $\sigma \in S_n$ and for any $k \in \mathbb{C}$ we have

   $$M_\sigma(k \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}) = M_\sigma \begin{pmatrix} k \\ \vdots \\ k \end{pmatrix} = \begin{pmatrix} k \\ \vdots \\ k \end{pmatrix} = k \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

   which means $k \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in W_{M_\sigma} \forall k \in \mathbb{C}$.

2. $W = W_{M_{(12\ldots n)}}$, as W is the +1-eigenspace of

   $$M_{(12\ldots n)} = \begin{pmatrix} e_2 & \Big| & e_3 & \Big| & \ldots & \Big| & e_n & \Big| & e_1 \end{pmatrix}.$$

3. If $W = W_{M_\sigma}$ then $|W_{M_\sigma}| = |W| = | < \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} > | = 1$. The other direction follows directly from 1.

□

Notes:

- Later we will see that $W = W_{M_\sigma}$ if and only if the permutation σ can be identified with a n-cycle.
- Using $W = W_{M_\sigma}$ for at least one $\sigma \in S_n$ we get the states of Proposition 1.1 and Corollary 1.2 also for W and V.

- Because of $\mathbb{C}^n = V \oplus W$ we find for every $x \in \mathbb{C}^n$ exactly one $v_x \in V$ and exactly one $w_x \in W$ with $x = v_x + w_x$ and get for all $\sigma \in S_n$ $f_\sigma(x) = f_\sigma(v_x+w_x) \underset{\substack{\uparrow \\ f_\sigma \text{ lin.}}}{=} f_\sigma(v_x)+f_\sigma(w_x) \underset{\substack{\uparrow \\ W \le W_{M_\sigma} \text{ 1-eigensp.}}}{=} f_\sigma(v_x)+w_x.$
  This shows, that it is enough to know, how f_σ behaves on $V \cong \mathbb{C}^n/W$. Instead of f_σ we could also regard the function

$$\begin{array}{rcl} f_\sigma/_W : \mathbb{C}^n/_W & \to & \mathbb{C}^n/_W \\ [x] & \mapsto & [f(x)] \end{array}$$

  without loss of generality.

3 A Partial Order on the Symmetric Group

Regard the length function

$$\begin{array}{rcl} l : S_n & \to & \{0, ..., n-1\} \\ \sigma & \mapsto & l(\sigma), \end{array}$$

giving the minimum amount of consecutive transpositions the permutation σ can be represented as.

Definition 3.1 (Brady, Reference [Bra01], Definition 2.1) For all $\tau, \sigma \in S_n$ we define

$$\tau \le \sigma :\Leftrightarrow l(\sigma) = l(\tau) + l(\tau^{-1}\sigma)$$

Lemma 3.2 *For all $\sigma \in S_n$ it is*

$$l(\sigma) = |V_{M_\sigma}|.$$

Proof. Let σ be any permutation of n elements. Without loss of generality we represent it as $\sigma = (p_{l(\sigma)}q_{l(\sigma)}) \dots (p_1 q_1)$ with $p_i \neq p_j \ \wedge \ q_i \neq q_j \ \forall i \neq j$ and if $q_j = p_i \Rightarrow j = i+1$, making up a shortest finite sequence of consecutive transpositions the permutation σ can be displayed as.

We regard $|V_{M_\sigma}| = rk(M_\sigma - I)$, where rk is the *column rank*.
We define $\sigma_0 := id$ and $\sigma_k := (p_k q_k) \dots (p_1 q_1)$ for any $k \in \{1, ..., l(\sigma)\}$. We show via induction that $rk(M_{\sigma_k} - I) = k$ for all $k \in \{0, ..., l(\sigma)\}$, especially for $k = l(\sigma)$.

1. Base Case k=0: $rk(M_{\sigma_0} - I) = rk(I - I) = 0$.

2. Induction Hypothesis: $rk(M_{\sigma_k} - I) = k$ for some $k \in \{0, ..., l(\sigma) - 1\}$.

3. Induction Step $k \to k+1$: We show $rk(M_{\sigma_{k+1}} - I) = rk(M_{\sigma_k} - I) + 1$.

   Let L be any linearly independent system of column vectors of $M_{\sigma_k} - I$ with $|L| = rk(M_{\sigma_k} - I)$. Then L is a basis for $im(M_{\sigma_k} - I) = V_{M_{\sigma_k}}$. We define $a := max\{\{1\} \cup \{i \in \{2, ..., k+1\} \big| q_i \neq p_{i-1}\}\}$.

   - If $a = k+1$, then $\sigma_{k+1} = (p_{k+1}q_a)\underbrace{(p_k q_k)...(p_1 q_1)}_{=\sigma_k}$ with $q_a \neq p_k$.

     We get

     $\to\ \sigma_k(p_i) = \sigma_{k+1}(p_i)\ \forall i \in \{1, ..., k\}$
     $\to\ \sigma_k(q_i) = \sigma_{k+1}(q_i)\ \forall i \in \{1, ..., k\}$
     $\to\ \sigma_k(p_{k+1}) = p_{k+1} \neq q_{k+1} = \sigma_{k+1}(p_{k+1})$
     $\to\ \sigma_k(q_{k+1}) = q_{k+1} \neq p_{k+1} = \sigma_{k+1}(q_{k+1})$.

   - If $a \leq k$, then we can display
     $\sigma_{k+1} = (p_{k+1}p_k...p_{a+1}p_a q_a)(p_{a-1}q_{a-1})...(p_1 q_1)$ and
     $\sigma_k = (p_k...p_{a+1}p_a q_a)(p_{a-1}q_{a-1})...(p_1 q_1)$ with $q_{i+1} = p_i$
     for all $i \in \{a, ..., k\}$. We get

     $\to\ \sigma_k(p_i) = \sigma_{k+1}(p_i)\ \forall i \in \{1, ..., k\}$
     $\to\ \sigma_k(q_i) = \sigma_{k+1}(q_i)\ \forall i \in \{1, ..., k+1\} \setminus \{a\}$
     $\to\ \sigma_k(p_{k+1}) = p_{k+1} \neq q_{k+1} = p_k = \sigma_{k+1}(p_{k+1})$
     $\to\ \sigma_k(q_a) = p_k = q_{k+1} \neq p_{k+1} = \sigma_{k+1}(q_a)$.

   This gives $\sigma_k(i) = \sigma_{k+1}(i)$ for all $i \in \{1, ..., n\} \setminus \{p_{k+1}, q_a\}$.

   Then, since $(M_{\sigma_k} - I)e_i = e_{\sigma_k(i)} - e_i$ and $(M_{\sigma_{k+1}} - I)e_i = e_{\sigma_{k+1}(i)} - e_i$ for all $i \in \{1, ..., n\}$, the following columns of the two matrices are equal:

   $$(M_{\sigma_k} - I)e_i \quad = \quad (M_{\sigma_{k+1}} - I)e_i \quad \forall i \in \{1, ..., n\} \setminus \{p_{k+1}, q_a\}.$$

   The other two columns are given by

   $$\begin{aligned} x_k &:= (M_{\sigma_k} - I)e_{p_{k+1}} = e_{p_{k+1}} - e_{p_{k+1}} = 0 \\ x_{k+1} &:= (M_{\sigma_{k+1}} - I)e_{p_{k+1}} = e_{q_{k+1}} - e_{p_{k+1}} \neq 0 \\ y_k &:= (M_{\sigma_k} - I)e_{q_a} = \begin{cases} e_{q_a} - e_{q_a} = 0, & \text{if } a = k+1 \\ e_{p_k} - e_{q_a} \neq 0, & \text{if } a \neq k+1 \end{cases} \\ y_{k+1} &:= (M_{\sigma_{k+1}} - I)e_{q_a} = e_{p_{k+1}} - e_{q_a} \neq 0 \end{aligned}$$

We get

$$\text{If } a = k+1 \Rightarrow x_{k+1} + y_{k+1} = 0 = y_k$$
$$\text{If } a \neq k+1 \Rightarrow q_{k+1} = p_k \Rightarrow x_{k+1} + y_{k+1} = y_k,$$

which gives $x_{k+1}+y_{k+1} = y_k$ in both cases. Together with the relation $x_k = 0 \notin L$ we find

$$L' := \begin{cases} L \cup \{x_{k+1}\}, & \text{if } y_k \notin L \\ (L \setminus \{y_k\}) \cup \{x_{k+1}, y_{k+1}\}, & \text{if } y_k \in L \end{cases}$$

as a maximal system of linearly independent column vectors of $M_{\sigma_{k+1}} - I$, which means a basis for $im(M_{\sigma_{k+1}} - I) = V_{M_{\sigma_{k+1}}}$:

- The system $L_1 := L \cup \{x_{k+1}\}$ is linearly independent in every case, because x_{k+1} has an entry on position p_{k+1}, which cannot be displayed as a linear combination of vectors of L: All vectors of L are column vectors of $M_{\sigma_k} - I$, and since $\sigma_k(p_{k+1}) = p_{k+1}$, the entry on position p_{k+1} is zero for all column vectors of $M_{\sigma_k} - I$.
- If $y_k \in L$, the system $L_2 := (L \setminus \{y_k\}) \cup \{x_{k+1}, y_{k+1}\}$ is linearly independent, since $y_{k+1} = y_k - x_{k+1}$ and L_1 is linearly independent.
- The system $L_3 := L \cup \{x_{k+1}, y_{k+1}\}$ is linearly dependent: The vectors x_{k+1} and y_{k+1} are both not in L, because each has an entry on position p_{k+1}. It is y_k either a vector of L or can be at least displayed as a linear combination of vectors of L, but also as a linear combination of x_{k+1} and y_{k+1} via $y_k = x_{k+1} + y_{k+1}$.

This means $|L'| = rk(M_{\sigma_{k+1}} - I)$. It is $|L_1| = |L| + 1$. If $y_k \in L$, then $|L_2| = (|L| - 1) + 2 = |L| + 1$. So we have $|L'| = |L| + 1$. Using the induction hypothesis, this gives the result.

□

Theorem 3.3 *Regard $f : S_n \hookrightarrow U(\mathbb{C}^n)$ the monomorphism defined in chapter 2. For all $\tau, \sigma \in S_n$:*

$$\tau \leq \sigma \Leftrightarrow M_\tau \leq M_\sigma$$

Proof. This theorem follows directly from Lemma 3.2 since

$$\tau \leq \sigma :\Leftrightarrow l(\sigma) = l(\tau) + l(\tau^{-1}\sigma) \Leftrightarrow |V_{M_\sigma}| = |V_{M_\tau}| + |V_{M_{\tau^{-1}\sigma}}| \Leftrightarrow: M_\tau \leq M_\sigma.$$

□

Lemma 3.4 (Brady, Reference [Bra01], Lemma 2.2)
For all $\tau = \tau_1 ... \tau_k \in S_n$ *with* k *disjoint cycles, it is* $l(\tau) = \sum_{i=1}^{k} l(\tau_k)$*.*

Proof. The proof given in [Bra01] is: "The disjoint cycles of τ can only be factored as products of transpositions using disjoint sets of transpositions". Note: Since the cycles are pairwise disjoint, the intersection $V_{\tau_i} \cap V_{\tau_j}$ is empty for $i \neq j$, and we have $V_{M_\tau} = \bigoplus_{i=1}^{k} V_{M_{\tau_i}}$, and especially $|V_{M_\tau}| = \sum_{i=1}^{k} |V_{M_{\tau_i}}|$. Lemma 3.2 gives $l(\tau) = \sum_{i=1}^{k} l(\tau_k)$.

□

Note: In [Bra01] Brady also says $\tau_i \leq \tau \forall i \in \{1, ..., k\}$. We show this in Lemma 3.8.

Lemma 3.5 (Brady, Reference [Bra01], Lemma 2.3)
A cycle c *of* S_n *permutating* k *distinct elements, with* $k \in \{1, ..., n\}$*, has length* $l(c) = k - 1$*.*

Proof. Regard some cycle $c = (x_1 ... x_k) \in S_n$ with some $k \in \{1, ..., n\}$ and for all $i_1, i_2 \in \{1, ..., k\}$: $i_1 \neq i_2 \Rightarrow x_{i_1} \neq x_{i_2}$, such that c permutes k distinct elements.

1. Then we can represent $c = (x_1 x_2)(x_2 x_3)...(x_{k-1} x_k)$, which gives $l(c) \leq k - 1$.

2. On the other hand we show $l(c) \geq k - 1$ via induction on k: Base case $k = 1$: $l(c) = l((x_k)) = 0 \geq 1 - 1 = k - 1$. Regard now $k > 1$. Induction Hypothesis: For all $t \in \{1, ..., k - 1\}$: A t-cycle has length $\geq t - 1$. Induction Step:

   (i) Regard $c = \tau_1 ... \tau_{l(c)}$ with transpositions $\tau_i = (p_i q_i)$ from S_n. Since $l(c)$ is the minimal amount of transpositions c can be displayed as, it is $\tau_i \neq \tau_j$ for $i \neq j$, and it is $\tau_1 c = \tau_1 \tau_1 \tau_2 ... \tau_{l(c)} = \tau_2 ... \tau_{l(c)}$ giving $l(\tau_1 c) \leq l(c) - 1$.

   (ii) Since $c = (x_1 ... x_k)$ moves elements $x_1, ..., x_k$, it must be $\tau_1 = (x_i x_j)$ with $i, j \in \{1, ..., k\}$, $i < j$ without loss of generality. This gives a representation

   $$\tau_1 c = (x_i x_j)(x_1 ... x_i ... x_j ... x_k) = (x_1 ... x_{i-1} x_j ... x_k)(x_i ... x_{j-1})$$

   and further by using step 1, Lemma 3.4 and the induction hypothesis:

   $$\begin{aligned} l(c) - 1 &\geq l(\tau_1 c) = l((x_1 ... x_{i-1} x_j ... x_k)) + l((x_i ... x_{j-1})) \\ &\geq ((i - 1) + (k - j) + 1) - 1 + ((j - 1) - i + 1) - 1 = k - 2. \end{aligned}$$

We get $l(c) - 1 \geq k - 2$ which means $l(c) \geq k - 1$.

Step 1. and 2. give the equality $l(c) = k - 1$.

□

Note: In the following we will claim for each cycle $c = (x_1...x_k) \in S_n$ that $i_1 \neq i_2 \Rightarrow x_{i_1} \neq x_{i_2}$ without loss of generality.

Lemma 3.6 (Brady, Reference [Bra01], Lemma 2.4)
If $\tau = c_1...c_k$ a permutation in S_n with k disjoint cycles (including 1-cycles), then it is $l(\tau) = n - k$.

Proof. Using Lemma 3.5 we get that for all $i \in \{1, ..., k\}$ we have $l(c_i) = l_i - 1$, where l_i is the amount of elements permuted by c_i. Since the cycles are disjoint and since we also regard 1-cycles, it is $\sum_{i=1}^k l_i = n$. Together with Lemma 3.4 we get $l(\tau) = \sum_{i=1}^k l(c_i) = \sum_{i=1}^k (l_i - 1) = (\sum_{i=1}^k l_i) - k = n - k$.

□

Definition 3.7 For any cycle $c_\tau = (x_1...x_{l(c_\tau)+1}) \in S_n$ and any permutation $\sigma \in S_n$:

1. We say that c_τ is contained in a cycle $c_\sigma = (y_1...y_{l(c_\sigma)+1})$ of σ ($c_\tau \subseteq c_\sigma$), iff
$$\forall i \in \{1, ..., l(c_\tau) + 1\} : \exists j(i) \in \{1, ..., l(c_\sigma) + 1\} : x_i = y_{j(i)}.$$
2. If c_τ is contained in c_σ, we say that c_τ *is ordered consistently with* σ, iff $\exists k \in \mathbb{N} : \forall i_1, i_2 \in \{1, ..., l(c_\tau) + 1\}$:
$$i_1 \leq i_2 \Rightarrow (j(i_1) + k)\ mod(l(c_\sigma) + 1) \leq (j(i_2) + k)\ mod(l(c_\sigma) + 1),$$
where the $\leq$ relation on the cosets is defined to be the comparison between their minimal nonnegative representatatives.

Lemma 3.8 *Regard any $\sigma \in S_n$ defined by disjoint cycles $c_1...c_k = \sigma$. Then $\prod_{i \in U} c_i \leq \sigma$ for all $U \subseteq \{1, ..., k\}$.*

Proof. Since disjoint cycles commute we have
$$\begin{aligned} l((\prod_{i \in U} c_i)^{-1}\sigma) &= l(\prod_{i \in \{1,...,k\} \setminus U} c_i) \\ &\overset{\text{Lemma 3.4}}{=} \sum_{i \in \{1,...,k\} \setminus U} l(c_i) = \sum_{i \in \{1,...,k\}} l(c_i) - \sum_{i \in U} l(c_i) \\ &\overset{\text{Lemma 3.4}}{=} l(\sigma) - l(\prod_{i \in U} c_i) \end{aligned}$$

which gives $\prod_{i \in U} c_i \leq \sigma$.

□

Proposition 3.9 (Brady, Reference [Bra01], Proposition 2.6) *Regard* $\sigma \in S_n$*. For all* $a, b \in \{1, ..., n\}$:

$$(ab) \leq \sigma \Leftrightarrow (ab) \textit{ is contained in one cycle of } \sigma.$$

Proof.

- "$\Rightarrow$": Let $(ab) \leq \sigma$. Suppose on the contrary that $c_a = (ax_1...x_{l(c_a)})$ is the cycle of σ containing a, and $c_b = (by_1...y_{l(c_b)})$ is the cycle of σ containing b. Let c_a be disjoint with c_b. Then

$$\begin{aligned}
l((ab)^{-1}c_a c_b) &= l((ab)c_a c_b) \\
&= l((ab)(ax_1...x_{l(c_a)+1})(by_1...y_{l(c_b)+1})) \\
&= l((ax_1...x_{l(c_a)+1}by_1...y_{l(c_b)+1}) \\
&\overset{\text{Lemma 3.5}}{=} (2 + (l(c_a) + 1) + (l(c_b) + 1)) - 1 \\
&= l(c_a) + l(c_b) - l((ab)) + 4 \neq l(\sigma) - l((ab)) \\
&\overset{\text{Lemma 3.4}}{=} l(c_a c_b) - l((ab)) + 4 \neq l(c_a c_b) - l((ab))
\end{aligned}$$

  giving $(ab) \not\leq c_a c_b$. Since $c_a c_b \leq \sigma$ by Lemma 3.8, transitivity gives $(ab) \not\leq \sigma$.

- "$\Leftarrow$": If (ab) is contained in cycle c_{ab} of σ, then $l((ab)^{-1}c_{ab}) = l((ab)(ax_1...x_t by_1...y_s)) = l((by_1...y_s)(ax_1...x_t)) = l(c_{ab}) - 1 = l(c_{ab}) - l((ab))$, giving $(ab) \leq c_{ab}$. Since $c_{ab} \leq \sigma$ by Lemma 3.8, transitivity gives $(ab) \leq \sigma$.

□

Lemma 3.10 *For any cycle* $c_\tau \in S_n$ *and any permutation* $\sigma \in S_n$*:*
We have $c_\tau \leq \sigma$ *if and only if*

(a) c_τ *is contained in a cycle of* σ *and*

(b) c_τ *is ordered consistently with* σ*.*

Proof.

- "$\Leftarrow$": We regard $\sigma = \sigma_1...\sigma_m$ disjoint cycles of σ (including 1-cycles). Since (a) holds for c_τ and σ, we find that there exists a cycle σ_s of σ such that c_τ is contained in σ_s and c_τ is disjoint with all other cycles of σ. We get

$$\begin{array}{rcl} {c_\tau}^{-1}\sigma_s & \stackrel{\text{Lemma 3.5}}{=} & (x_1...x_{l(c_\tau)+1})^{-1}(y_1...y_{l(\sigma_s)+1}) \\ & = & (x_{l(c_\tau)+1}...x_1)(y_1...y_{l(\sigma_s)+1}) \\ & \stackrel{(a)}{=} & (y_{j(l(c_\tau)+1}...y_{j(2)}y_{j(1)})(y_1...y_{l(\sigma_s)+1}) \\ & \stackrel{(b)}{=} & (y_{j(l(c_\tau)+1)}...y_{j(2)}y_{j(1)})(y_1...y_{j(1)}...y_{j(2)}...y_{j(l(c_\tau)+1)}...y_{l(\sigma_s)+1}) \\ & = & \underbrace{(y_{j(1)}...y_{j(2)-1})}_{=:c_1}\underbrace{(y_{j(2)}...y_{j(3)-1})}_{=:c_2}... \\ & & \quad ...\underbrace{(y_{j(l(c_\tau)+1)}...y_{l(\sigma_s)+1}y_1...y_{j(1)-1})}_{=:c_{l(c_\tau)+1}} \end{array}$$

and further since disjoint cycles commute

$$\begin{array}{rcl} c_\tau^{-1}\sigma & = & c_\tau^{-1}\sigma_1...\sigma_m \\ & = & \sigma_1...\sigma_{s-1}c_\tau^{-1}\sigma_s\sigma_{s+1}...\sigma_m \\ & = & \sigma_1...\sigma_{s-1}c_1...c_{l(c_\tau)+1}\sigma_{s+1}...\sigma_m \\ & =: & \tilde{c}_1...\tilde{c}_p \end{array}$$

giving us $p := (m-1) + (l(c_\tau)+1)$ disjoint cycles. Using Lemma 3.6 we get

$$\begin{array}{rcl} l(c_\tau^{-1}\sigma) & = & n - p \\ & = & n - ((m-1) + (l(c_\tau)+1)) \\ & = & n - (n - l(\sigma) - 1 + l(c_\tau) + 1) \\ & = & l(\sigma) - l(c_\tau) \end{array}$$

which gives $c_\tau \leq \sigma$.

- "$c_\tau \leq \sigma \Rightarrow (a)$": (Brady, Reference [Bra01], Proposition 2.7) Suppose not. Then there exist two elements $a, b \in \{1, ..., n\}$ contained in c_τ but not contained in one cycle of σ. Then using Proposition 3.9 we get $(ab) \leq c_\tau$ and $(ab) \not\leq \sigma$. Transititity gives $c_\tau \not\leq \sigma$.

- "$c_\tau \leq \sigma \Rightarrow (b)$": Suppose not.

$$\begin{array}{rcl} {c_\tau}^{-1}\sigma_s & \stackrel{\text{Lemma 3.5}}{=} & (x_1...x_{l(c_\tau)+1})^{-1}(y_1...y_{l(\sigma_s)+1}) \\ & = & (x_{l(c_\tau)+1}...x_1)(y_1...y_{l(\sigma_s)+1}) \\ & \stackrel{(a)}{=} & (y_{j(l(c_\tau)+1}...y_{j(2)}y_{j(1)})(y_1...y_{l(\sigma_s)+1}) \\ & \stackrel{\neg(b)}{=} & (y_{j(l(c_\tau)+1)}...y_{j(2)}y_{j(1)}) \\ & & \quad (y_1...y_{j(\psi(1))}...y_{j(\psi(2))}...y_{j(\psi(l(c_\tau)+1))}...y_{l(\sigma_s)+1}) \end{array}$$

with some permutation $\psi \in S_{l(c_\tau)+1}$, $\psi \neq id$. This gives less than $l(c_\tau)+1$ disjoint cycles, and with Lemma 3.6 we get $l(c_\tau^{-1}\sigma) \neq l(\sigma) - l(c_\tau)$, which means $c_\tau \not\leq \sigma$.

□

The definition of a cycle ordered consistently with a permutation in this thesis is different from the Definition 2.8 of Brady, Reference [Bra01]. We now show that both definitions are equivalent:

Lemma 3.11 *For any cycle $c_\tau \in S_n$ with $l(c_\tau) > 1$ and for any permutation $\sigma \in S_n$:*

$$\begin{aligned} & c_\tau \text{ is ordered consistently with } \sigma \\ \Leftrightarrow \quad & \forall a,b,c \in \{1,...,n\} : (abc) \leq c_\tau \Rightarrow (abc) \leq \sigma \end{aligned}$$

Proof. Regard $c_\tau = (x_1 ... x_{l(c_\tau)+1})$. Define c_σ as the cycle of σ containing the element x_1. Without loss of generality we represent it as $c_\sigma = (y_1 ... y_{l(c_\sigma)+1})$ with $y_1 = x_1$. Using Lemma 3.10 we get $\forall i_1, i_2 \in \{1,...,l(c_\tau)+1\}$, $i_1 \neq i_2$:

$$\begin{aligned} (x_1 x_{i_1} x_{i_2}) \leq c_\tau \quad & \Leftrightarrow \quad i_1 < i_2 \\ \Downarrow \quad & \\ (x_1 x_{i_1} x_{i_2}) \leq \sigma \quad & \Leftrightarrow \quad \exists j(i_1), j(i_2) \in \{1,...,l(c_\sigma)+1\} : j(i_1) < j(i_2) \\ & \qquad \text{and } x_{i_1} = y_j(i_1) \text{ , } x_{i_2} = y_{j(i_2)} \end{aligned}$$

Since x_1 could be chosen as any element of c_τ by using the modulo operation, the result follows by setting $a = x_1, b = x_{i_1}$ and $c = x_{i_2}$.

□

Now, since we have characterized the meaning of a cycle being lower than or equal to a permutation, we can define the following:

Definition 3.12 (Brady, Reference[Bra01], Definition 2.11) Regard $\tau, \sigma \in S_n$. We say that τ *has crossing cycles with respect to* σ, iff

$\exists a,b,c,d \in \{1,...,n\} : (abcd) \leq \sigma$ and $(ac) \leq \tau$ and $(bd) \leq \tau$ but $(abcd) \not\leq \tau$.

Lemma 3.13 *Let $(\tau_i)_{i \in \{1,...,k\}}$ be any family of pairwise disjoint cycles of S_n, and $(\sigma_i)_{i \in \{1,...,k\}}$ be any family of pairwise disjoint cycles of S_n. Then*

$$\tau_i \leq \sigma_i \forall i \in \{1,...,k\} \Rightarrow \prod_{i=1}^{k} \tau_i \leq \prod_{i=1}^{k} \sigma_i.$$

Proof. Lemma 3.10 gives, that for all $j \in \{1, ..., k\}$ τ_j is contained in σ_j and disjoint with all other cycles of $(\sigma_i)_{i \in \{1,...,k\} \setminus \{j\}}$. This gives

$$\begin{aligned} l\Big((\prod_{i=1}^{k} \tau_i)^{-1} \prod_{i=1}^{k} \sigma_i\Big) &= l\Big(\prod_{i=1}^{k} \tau_i^{-1} \sigma_i\Big) \\ &\overset{\text{Lemma 3.4}}{=} \sum_{i=1}^{k} l(\tau_i^{-1} \sigma_i) \\ &\overset{\tau_i \le \sigma_i \forall i \in \{1,...,k\}}{=} \sum_{i=1}^{k} \big(l(\sigma_i) - l(\tau_i)\big) \\ &= \sum_{i=1}^{k} l(\sigma_i) - \sum_{i=1}^{k} l(\tau_i) \\ &\overset{\text{Lemma 3.4}}{=} l\Big(\prod_{i=1}^{k} \sigma_i\Big) - l\Big(\prod_{i=1}^{k} \tau_i\Big) \end{aligned}$$

which gives the result.

□

Lemma 3.14 (Brady, Reference [Bra01], Lemma 2.12)
For all distinct $a, b, c, d \in \{1, ..., n\}$ we have $(ac)(bd) \not\le (abcd)$.

Proof. $l\Big(\big((ac)(bd)\big)^{-1}(abcd)\Big) = l\Big((bd)\big((ac)(abcd)\big)\Big) = l\big((bd)(ab)(cd)\big) = l\big((adcb)\big) = 3 \neq 1 = 3 - 2 = l\big((abcd)\big) - l\big((ac)(bd)\big)$, which gives the result.

□

Thoerem 3.15 (Brady, Reference [Bra01], Theorem 2.14)
Regard any $\tau, \sigma \in S_n$. We have $\tau \le \sigma$ if and only if

(a) each cycle of τ is contained in a cycle of σ and

(b) each cycle of τ is ordered consistently with σ and

(c) τ has no crossing cycles with respect to σ.

Proof.

- $(a), (b)\&(c) \Rightarrow \tau \le \sigma$: We show via induction on the amount t of distinct elements transposed by σ, that $l(\tau^{-1}\sigma) = l(\sigma) - l(\tau)$ for all $\tau \in S_n$ fulfilling (a), (b) and (c) with σ.

  1. Base Case $t = 0$: Then $\sigma = id$ and because of (a) it is also $\tau = id$. This gives $l(\tau^{-1}\sigma) = l(id) = 0 = l(id) - l(id) = l(\sigma) - l(\tau)$. Regard now $t > 0$:

2. Induction Hypothesis: For all $\sigma \in S_n$ transposing less than t distinct elements it is $l(\tau^{-1}\sigma) = l(\sigma) - l(\tau)$ for all $\tau \in S_n$ fulfilling (a), (b) and (c) with σ.

3. Induction Step: If σ consists of more than one cycle, each cycle of τ^{-1} can commute with those cycles of σ it is disjoint with. Then we can show the result directly from the induction hypothesis by using Lemma 3.4. So we regard just one cycle $\sigma = c_\sigma$. Let $\tau = \tau_1 ... \tau_k$ be disjoint cycles of τ. Because disjoint cycles commute and because the result is trivial for $\tau = id$, we can assume that $\tau_1 \neq id$. Since (a) and (b) hold for τ and σ, the proof of Lemma 3.10 shows that for $\tau_1 = (x_1 ... x_{l(\tau_1)+1})$ and for $\sigma = (y_1 ... y_{l(\sigma)+1})$ we have

$$\tau_1^{-1}\sigma = \underbrace{(y_{j(1)} \cdots y_{j(2)-1})}_{=:c_1} \underbrace{(y_{j(2)} \cdots y_{j(3)-1})}_{=:c_2} \cdots$$
$$\cdots \underbrace{(y_{j(l(\tau_1)+1)} \cdots y_{l(\sigma)+1} y_1 \cdots y_{j(1)-1})}_{=:c_{l(\tau_1)+1}}$$

and $\tau_1 \leq \sigma$.
We define $\tilde{\sigma} := \tau_1^{-1}\sigma$ and $\tilde{\tau} := \tau_1^{-1}\tau$ and get

$$\tau^{-1}\sigma = \tilde{\tau}^{-1}\tilde{\sigma}.$$

Now we show the conditions (a),(b) and (c) for $\tilde{\tau}$ and $\tilde{\sigma}$:

(a) Each cycle of $\tilde{\tau}$ is contained in a cycle of $\tilde{\sigma}$: Suppose on the contrary that there exists $r \in \{2, ..., k\}$ such that τ_r contains two Elements y_u and y_v contained in different cycles of $\tilde{\sigma}$, with indices $u, v \in \{1, ..., l(\sigma) + 1\}$, $u \neq v$. Because τ_r and τ_1 are disjoint, this means that there exist two indices $i_1, i_2 \in \{1, ..., l(\sigma_s) + 1\}$, $i_1 \neq i_2$, such that $j(i_1) < u < j(i_2) < v$ or $u < j(i_1) < v < j(i_2)$.
We regard the first case. Because of Lemma 3.10 we would find $(y_{j(i_1)} y_u y_{j(i_2)} y_v) \leq \sigma$, $(y_u y_v) \leq \tau_r \leq \tau$ and $(y_{j(i_1)} y_{j(i_2)}) \leq \tau_1 \leq \tau$, but $(y_{j(i_1)} y_u y_{j(i_2)} y_v) \not\leq \tau$, which would led to the wrong state that τ has crossing cycles with respect to σ.
In the other case we would get an analogue contradiction.

(b) Each cycle of $\tilde{\tau}$ is ordered consistently with $\tilde{\sigma}$, because each cycle of $\tilde{\tau}$ is also a cycle of τ, because (b) holds for τ and σ, and because we can represent $\tilde{\sigma}$ and σ in a way that the order of elements contained in the cycles of $\tilde{\sigma}$ is the same than in σ.

(c) The Permutation $\tilde{\tau}$ has no crossing cycles with respect to $\tilde{\sigma}$: Suppose there exist $a, b, c, d \in \{1, ..., n\} : (abcd) \leq \tilde{\sigma}$ and $(ac) \leq \tilde{\tau}$ and $(bd) \leq \tilde{\tau}$. We show $(abcd) \leq \tilde{\tau}$. Using Lemma 3.10 we get $(abcd) \leq \sigma$ and $(ac) \leq \tau$ and $(bd) \leq \tau$. Since τ has no crossing cycles with respect to σ, it follows $(abcd) \leq \tau$ and further $(abcd) \leq \tilde{\tau}$, because $(abcd)$ is disjoint with τ_1.

Because (a), (b) and (c) also hold for $\tilde{\tau}$ and $\tilde{\sigma}$, we can use the induction hypothesis for each cycle of $\tilde{\sigma}$ by using Lemma 3.4 after having commuted the cycles of $\tilde{\tau}$ with those of $\tilde{\sigma}$ they are disjoint with. We get

$$\begin{aligned} l(\tau^{-1}\sigma) &= l(\tilde{\tau}^{-1}\tilde{\sigma}) \\ &= l(\tilde{\sigma}) - l(\tilde{\tau}) \end{aligned}$$

Since $\tau_1 \leq \sigma$ by Lemma 3.10 and $\tau_1 \leq \tau$ by Lemma 3.8 this is equal to

$$\begin{aligned} & l(\sigma) - l(\tau_1) - (l(\tau) - l(\tau_1)) \\ = \; & l(\sigma) - l(\tau), \end{aligned}$$

which gives $\tau \leq \sigma$.

- $\tau \leq \sigma \Rightarrow (a), (b)$: For each cycle c_τ of τ: Lemma 3.8 gives $c_\tau \leq \sigma$. Then Lemma 3.10 gives the states (a) and (b).

- $\tau \leq \sigma \Rightarrow (c)$ (Brady, Reference[Bra01], Proposition 2.13): Suppose there exist $a, b, c, d \in \{1, ..., n\}$ such that $(abcd) \leq \sigma$, $(ac) \leq \tau$ and $(bd) \leq \tau$. We show $(abcd) \leq \tau$. Since $(abcd) \leq \sigma$, Lemma 3.10 gives that a, b, c and d are contained in the same cycle of σ, and $(acbd) \not\leq \sigma$. Let further τ_{ac} be the cycle of τ containing a and c, and τ_{bd} be the cycle of τ containing b and d.

  - If $\tau_{ac} = \tau_{bd}$, then we find a representation such that either $\tau_{ac} = (...a...b...c...d...)$ or $\tau_{ac} = (...a...c...b...d...)$. If the second case holded, Lemma 3.8 would give $\tau_{ac} \leq \tau$ and with Lemma 3.10 and transitivity we would get $(acbd) \leq \tau_{ac} \leq \tau \leq \sigma$ in contradiction to $(acbd) \not\leq \sigma$. So it is $\tau_{ac} = (...a...b...c...d...)$, and Lemma 3.8 and Lemma 3.10 give $(abcd) \leq \tau$.

  - If $\tau_{ac} \neq \tau_{bd}$, then since we assumed them to be disjoint, Lemma 3.13 gives $(ac)(bd) \leq \tau_{ac}\tau_{bd}$. With Lemma 3.8 we get $\tau_{ac}\tau_{bd} \leq \tau$ and transitivity gives $(ac)(bd) \leq \sigma$. Lemma 3.14 now gives the contradiction $(ac)(bd) \not\leq (abcd) \leq \sigma$.

□

Lemma 3.16 (Brady, Reference [Bra01], Lemma 3.9) *For $\tau, \sigma \in S_n$:*

$$\begin{aligned} \tau \leq \sigma \quad &\Leftrightarrow \quad \tau^{-1}\sigma \leq \sigma \\ &\Leftrightarrow \quad \sigma\tau^{-1} \leq \sigma. \end{aligned}$$

Proof. Lemma 1.8 gives the result for the unitary group, and the result for S_n follows by Theorem 3.3.

□

Lemma 3.17 (Brady, Reference [Bra01], Lemma 3.10) *For $\tau, \sigma, \phi \in S_n$, if $\phi \leq \tau \leq \sigma$, then $\phi^{-1}\tau \leq \phi^{-1}\sigma$ and $\tau^{-1}\sigma \leq \phi^{-1}\sigma$.*

Proof. Lemma 1.9 gives the result for the unitary group, and the result for S_n follows with Theorem 3.3.

□

Note: Brady's proof of his Lemma 3.9 in [Bra01] is a bit different from our proof of our Lemma 1.8. This might help us to understand that the partial order can indeed be transferred from the unitary group to the symmetric group.

3.1 Example S_n with $n = 3$

$S_3 = \{id, (12), (23), (13), (123), (132)\}$

$$M_{id} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I \ , \ M_{(123)} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \ , \ M_{(132)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \ ,$$

$$M_{(12)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \ M_{(23)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \ , \ M_{(13)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$|V_{M_{id}}| = |im(M_{id} - I)| = |im \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}| = 0 = l(id)$$

$$|V_{M_{(12)}}| = |im(M_{(12)} - I)| = |im \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}| = 1 = l((12))$$

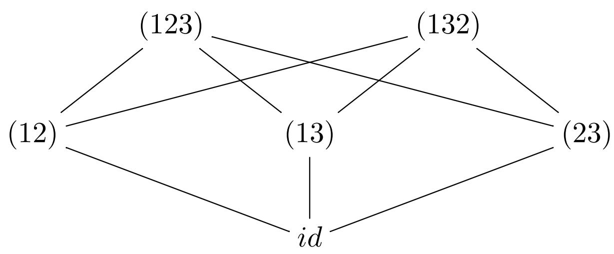
$$|V_{M_{(23)}}| = |im(M_{(23)} - I)| = |im \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}| = 1 = l((23))$$

$$|V_{M_{(13)}}| = |im(M_{(13)} - I)| = |im \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}| = 1 = l((13))$$

$$|V_{M_{(123)}}| = |im(M_{(12)} - I)| = |im \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} | = 2 = l((123))$$

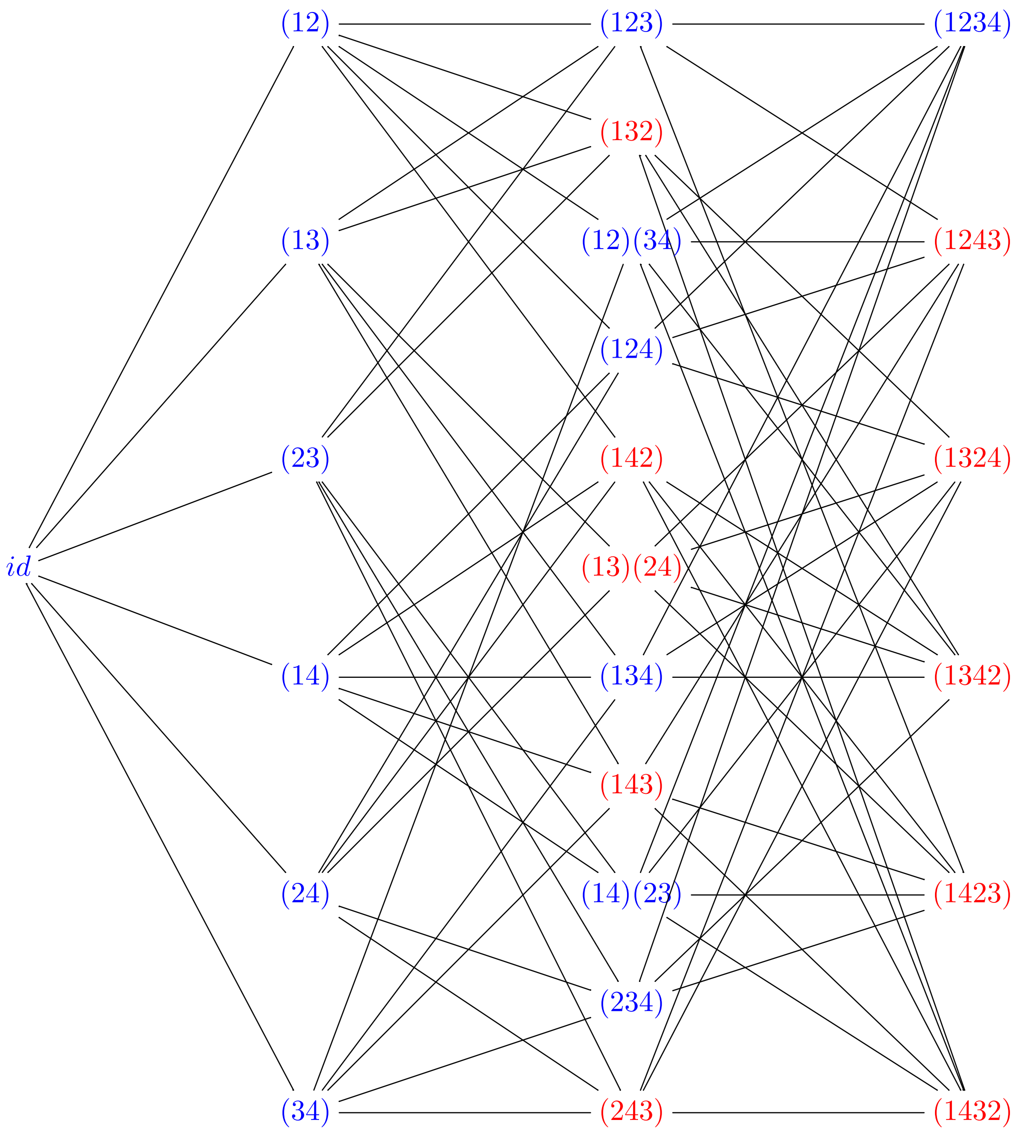
$$|V_{M_{(132)}}| = |im(M_{(132)} - I)| = |im \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} | = 2 = l((132))$$

We get the following partial order on S_3:



3.2 Example S_n with $n = 4$

We get the following partial order on S_4:



The elements displayed in blue color are the "allowable elements", the topic of the next chapter:

4 The Lattice of Allowable Elements

Since we have studied the set of permutations as a partially ordered set, we now ask for a possibility to form a lattice.

We now regard the set Π_n of all partitions of $\{1, ..., n\}$. It is a partially ordered set $(\Pi_n, \leq)$ with the partial order $\leq$ defined via $\pi_1 \leq \pi_2$, iff every block of π_1 is contained in a block of π_2 (Reference [Sta12]). The cycle structure of a permutation $\sigma \in S_n$ defines a partition of Π_n. We will denote this partition $\{\sigma\}$ like Brady did (Reference [Bra01]).

Definition 4.1 (Brady, Reference [Bra01], Definition 3.1) We fix the n-cycle $\gamma := (1...n)$. Then we define

$$\mathscr{A} := \{\sigma \in S_n \big| \sigma \leq \gamma\}$$

the set of "allowable elements".

Definition 4.2 (Brady, Reference [Bra01]) Regard any partition $P \in \Pi_n$.

- We say that two blocks B_1 and B_2 of P *cross*, iff there are integers a, b, c and $d \in \{1, ..., n\}$ so that $1 \leq a < b < c < d \leq n$ and $a, c \in B_1$ and $b, d \in B_2$.
- P is called noncrossing if none of its blocks cross.

Like in [Bra01] we will denote $NCP(n)$ the subset of noncrossing partitions of Π_n. For $h : \mathscr{A} \to \Pi_n,\ \sigma \mapsto \{\sigma\}$ it is $im(h) = NCP(n)$.

Lemma 4.2 (Brady, Reference [Bra01], Lemma 3.2) *For all* $\tau, \sigma \in \mathscr{A}$:

$$\tau \leq \sigma \Leftrightarrow \{\tau\} \leq \{\sigma\}$$

Proof. Since $\tau \leq \gamma$ and $\sigma \leq \gamma$ Theorem 3.15 gives that the permutation τ is ordered consistently with σ and that τ has no crossing cycles with respect to σ. So Theorem 3.15 gives $\tau \leq \sigma \Leftrightarrow$ each cycle of τ is contained in a cycle of σ, which is equivalent to $\{\tau\} \leq \{\sigma\}$.

□

In Reference [Kre72] Krevaras shows that the poset $NCP(n)$ forms a lattice. In [Bra01] it says "The meet operation is made up by intersection and the join operation is made up by the noncrossing closure of the union." In [Bra01] Brady also mentions that "the rank function on $NCP(n)$ given in [Kre72] corresponds with the length function on $\mathscr{A}$", and that "the atoms correspond to the transpositions".

Definition 4.3 (Brady, Reference [Bra01], Definition 3.3) "For each $\tau, \sigma \in \mathscr{A}$ we define the permutation $\tau \wedge \sigma$ by the following conditions.

(a) The numbers x and y belong to the same cycle of $\tau \wedge \sigma$ if and only if x and y belong to the same cycles in both τ and σ.

(b) The order of elements in the cycles of $\tau \wedge \sigma$ is consistent with γ."

Lemma 4.4 (Brady, Reference [Bra01], Lemma 3.4) *For each $\tau, \sigma \in \mathscr{A}$: $\tau \wedge \sigma \in \mathscr{A}$ and $\tau \wedge \sigma$ is the greatest lower bound of τ and σ. We call it* meet *of τ and σ.*

Proof. Using Theorem 2 of reference [Kre72] we get that $\{\tau \wedge \sigma\} \in NCP(n)$. Further it is $\{\tau \wedge \sigma\} = \{\tau\} \wedge \{\sigma\}$. We get that $\tau \wedge \sigma \in \mathscr{A}$. The result follows from Lemma 4.2.

□

Definition 4.5 (Brady, Reference [Bra01], Definition 3.5) "For each $\tau, \sigma \in \mathscr{A}$: We define the permutation $\tau \vee \sigma$ by the following conditions.

(a) $\{\tau \vee \sigma\} = \{\tau\} \vee \{\sigma\}$.

(b) The order of elements in the cycles of $\tau \vee \sigma$ is consistent with γ."

Lemma 4.6 (Brady, Reference [Bra01], Lemma 3.6) *For each $\tau, \sigma \in S_n$: $\tau \vee \sigma \in \mathscr{A}$ and $\tau \vee \sigma$ is the least upper bound of τ and σ. We call it* join *of τ and σ.*

Proof. Because $\{\tau\} \vee \{\sigma\} \in NCP(n)$ and because the cycles of $\{\tau\} \vee \{\sigma\}$ are ordered consistently with γ, it is $\tau \vee \sigma \in \mathscr{A}$. Using Theorem 3 of [9], $\{\tau\} \vee \{\sigma\}$ is the least upper bound of $\{\tau\}$ and $\{\sigma\}$ in $NCP(n)$, and Lemma 4.2 gives the result for $\mathscr{A}$.

□

Now we have got the following theorem:

Theorem 4.7 (Brady, Reference [Bra01], Theorem 3.7) *The poset $(\mathscr{A}, \leq)$ with the above definitions of meet and join forms a lattice.*

In chapter 3 we saw that the symmetric group can be regarded as a subgroup of the unitary group over a finite dimensional unitary vector space. It is also possible to generalize the theorem above: Two distinct elements of $U(\mathbb{C}^n)$ do not have a common upper bound. Because of that the poset $U(\mathbb{C}^n)$ does not form a lattice by itself. So we regard an interval of elements of the unitary group. The basic idea is the same as using allowable elements:

Theorem 4.8 (Brady and Watt, Reference [BW02], Theorem 2) *If $A \leq C$ in $U(\mathbb{C}^n)$ and $|V_C| - |V_A| = m \in \{0, ..., n\}$, then the interval $[A, C] = \{B \in U(\mathbb{C}^n) | A \leq B \leq C\}$ is isomorphic to the lattice of subspaces of $\mathbb{C}^m$ under inclusion.*

Proof. Since the interval $[V_A, V_C]$ in the lattice of subspaces of $\mathbb{C}^n$ has dimension $|V_C| - |V_A| = m$, it is isomorphic to $\mathbb{C}^m$.

Regard $g : [A, C] \to [V_A, V_C], B \mapsto V_B$:

- The function g is well defined: Let $B \in [A, C]$. Then Corollary 1.7 gives

$$\left.\begin{array}{l} A \leq B \Rightarrow V_A \subseteq V_B \\ B \leq C \Rightarrow V_B \subseteq V_C \end{array}\right\} \Rightarrow V_B \in [V_A, V_C]$$

- The function g is bijective: Regard any $S \in [V_A, V_C]$. We show that there exists exactly one $B \in [A, C]$ such that $V_B = S$.

  (a) Theorem 1.12 gives $S \subseteq V_C \Rightarrow \exists! B \in U(\mathbb{C}^n) : B \leq C \wedge V_B = S$.

  (b) Since $V_A \subseteq V_B$ Theorem 1.12 gives that $\exists! A' \in U(\mathbb{C}^n)$: $A' \leq B \wedge V_{A'} = V_A$. Since $B \leq C$ by (a), transitivity gives $A' \leq C$. Since $V_A \subseteq V_C$, the uniqueness part gives $A = A'$ and then we have $A \leq B$.

  The result follows from (a) and (b).

- The function g respects the partial orders: Regard $A \leq B \leq B' \leq C$. Then by Corollary 1.7: $V_A \subseteq V_B \subseteq V_{B'} \subseteq V_C$.

- The function g^{-1} respects the partial orders: Regard $V_A \subseteq U \subseteq U' \subseteq V_C$. Since g is bijective we find unique $B, B' \in [A, C]$ such that $U = V_B$ and $U' = V_{B'}$. We regard a restriction of g mapping bijectively from the interval $[A, B']$ to the interval $[V_A, V_{B'}]$, and find that for $U \in [V_A, V_{B'}]$ there is a unique $B'' \in [A, B']$ satisfying $U = V_{B''}$. Because of $[A, B'] \subseteq [A, C]$ we get $B'' = B$ and because of $B'' \leq B'$ we get $B \leq B'$.

□

Corollary 4.9 (Brady and Watt, Reference [BW02], Corollary 4) *If $C \in U(\mathbb{C}^n)$ and $S_1 \subset S_2 \subset ... \subset S_k = V_C$ is a chain of subspaces in V_C, then C factors uniquely as a product of k transformations $C = B_1 B_2 ... B_k$ with $B_1 B_2 ... B_i \leq C$ and $V_{B_1 B_2 ... B_i} = S_i$. for all $i \in \{1, ..., k\}$.*

Proof. For any $i \in \{1, ..., k\}$ it is $S_i \subseteq V_C$, so Theorem 1.12 gives a unique $C_i \in U(\mathbb{C}^n)$ such that $C_i \leq C$ and $V_{C_i} = S_i$. We define $B_1 := C_1$ and $B_i = (C_{i-1})^{-1} C_i$ for $i \in \{2, ..., k\}$, so that $B_1 B_2 ... B_i \leq C$ and $V_{B_1 B_2 ... B_i} = S_i$. By Theorem 1.12 it is B_1 unique. It follows via induction that B_i is unique for all $i \in \{1, ..., k\}$.

□

Using this Corollary in case of maximal chains, we get a strong version of the Cartan-Dieudonné Theorem, with respect to the fact that $char(\mathbb{C}) \neq 2$:

Corollary 4.10 (Brady and Watt, Reference [BW02], Corollary 5) *If* $C \in U(\mathbb{C}^n)$ *with* $|V_C| = k$ *and* $\{0\} \subset S_1 \subset S_2 \subset ... \subset S_k = V_C$ *is a maximal flag in* V_C*, then* C *factors uniquely as a product of* k *complex reflections,* $C = R_1R_2...R_k$*, with* $V_{R_1R_2...R_i} = S_i$ *for all* $i \in \{1, ..., k\}$*.*

Proof. For all $i \in \{1, ..., k\}$ regard C_i and B_i defined in the proof of Corollary 4.9. We further define $S_0 := \{0\} \subset V_C$, then it is $C_0 := I$ the unique transformation satisfying $C_0 \leq C$ and $V_{C_0} = S_0$.
It is $S_i = V_{C_i}$ and $C_i \in [I, C]$ for all $i \in \{0, .., k\}$. Since the function g^{-1} defined in the proof of Theorem 4.8 respects the partial orders, we get for all $i \in \{1, ..., k\} : V_{C_{i-1}} \subset V_{C_i} \Rightarrow C_{i-1} \leq C_i$. This gives

$$\begin{aligned} |V_{B_i}| &= |V_{(C_{i-1})^{-1}C_i}| \\ &\underset{C_{i-1} \leq C_i}{\underset{\uparrow}{=}} |V_{C_i}| - |V_{C_{i-1}}| \\ &= |S_i| - |S_{i-1}| \underset{\text{max. flag}}{\underset{\uparrow}{=}} 1. \end{aligned}$$

Using Corollary 1.14 we get that B_i is a *complex* reflection, so we set $R_i := B_i$ for all $i \in \{1, ..., k\}$.

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